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PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVE--ETC(U)
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FOREIGN TECHNOLOGY DIVISION

PROBLEMS OF DIFFRACTION AND PROPAGATION OF
ELECTROMAGNETIC WAVES
(Selected Pages)

by

V.A. Fok

FTD-ID(RS)T-0597-82

UNEDITED MACHINE TRANSLATION

FTD-ID(RS)T-0597-82 2 August 1982

MICROFICHE NR: FTD-82-C-001046

PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVES (Selected Pages)

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English pages: 205

Source: Problemy Diffraktsii i Rasprostraneniya Elektromagnitnykh Voln, Publishing House "Sovetskoye Radio, Moscow, 1970, pp. 383-517

Country of origin: USSR
This document is a machine translation.
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PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

Table of Contents

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

GRAPHICS DISCLAIMER

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PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVES.

V. A. Fok.

Page 383.

Chapter 18.

The theory of the coastal refraction of electromagnetic waves [1].

FOOTNOTE [1]. Greenberg and Fok, 1948. In this edition the text is somewhat abbreviated/reduced. ENDFOOTNOTE.

The problem is examined about the propagation of the electromagnetic waves above the infinite plane, which consists of two or more number of parts with the dissimilar electrical properties (sea and dry land, or sea and island). For the value of the vertical component of electric field in air in plane itself is derived integral equation. Subsequently is examined the case of the unlimited rectilinear shore, and equation is written out in the explicit form for this case. This is an integral second-order equation with the semi-infinite limits and the kernel, which depends on the absolute value of a difference in the arguments. To it is applied the mathematical theory, developed by Foch in the work of 1944 (see addition 1). As a result is obtained the explicitly strict solution of integral equation. Then is investigated the approximate form of solution, applied in the case of not too oblique incidence in the

wave.

1. Introduction.

While a question about the propagation of electromagnetic waves from the arbitrary emitter in the presence in space of two different homogeneous isotropic media with the flat surface of section can be at the present time be considered as the completely solved because of the works of Sommerfield and other authors, the corresponding more complex problem, which relates to case of three or more number of media, thus far it is still little investigated. A similar research would be, however, of considerable both theoretical and practical interest. In particular, the question about the coastal refraction, which concerns the incidence of electromagnetic waves to the coast and their reflection from it, is reduced (schematically) and to the examination of the case of three different media (air, sea, dry land), indicated in Fig. 1.

Page 384.

In this case, even if for simplification in the task of considering sea as ideal conductor and to count surface of OA (Fig. 1) of the section of sea and earth/ground of flat/plane and completely sharp, and even then it is necessary to consider not only

the properties of soil in the region of dry land, but also that angle which composes the boundary plane OA with the surface of sea.

In view of the difficulty of task let us introduce into its setting such simplifications which, being from the physical side sufficiently justified, would afford at the same time the possibility to bring its solution to the end/lead.

As the basis of this simplified treatment of a question it is possible to place according to Leontovich [21] the approximate boundary conditions for E and H (see Chapter 11). Specifically,, Leontovich showed that on the interface of two different media 1 and 2, from which the second possesses the much best conductivity, than the first, the tangential components of electrical and magnetic field in the first medium approximately satisfy on the very interface the relationships/ratios

$$\left.\begin{aligned} E_x &= \sqrt{\frac{\mu_2}{\eta_2}}\, H_y, \\ E_y &= -\sqrt{\frac{\mu_2}{\eta_2}}\, H_x, \end{aligned}\right\} \qquad (1.01)$$

where μ_2 - magnetic permeability of the second medium; η_2 - its complex dielectric constant whose modulus/module is considered large in comparison with unity. (These relationships/ratios they are obtained from the relationships/ratios (3.02) of Chapter 5 when $n_x = 0,\ n_y = 0,\ n_z = -1$.)

Relationships/ratios (1.01) are valid not only for the flat surface of section, but also for the arbitrary surface when smallest of the radii of curvature is great in comparison with the thick layer of skin-effect. They are obtained from that consideration, that with the sufficiently large conductivity of the second medium the character of the decrease of field with the deepening into it (exponential decay) virtually does not depend on the character of field on the interface of media. Formulas (1.01) can be considered as the approximate boundary conditions which must satisfy field in the first medium on the interface of media 1 and 2; they give the possibility to separate/liberate the solution of the problem about the field in the first medium from a question about field distribution in the second.

This approximate treatment of task will be, as a rule, justified when the thickness of skin-layer in the second medium at the frequency in question will be small in comparison with the wavelength λ in the first medium and when the properties of conducting layer relatively slowly are changed from one point of interface to another.

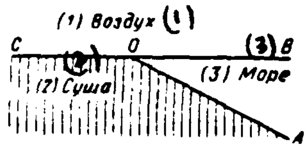

Fig. 1. Vertical section of coast.

Key: (1). Air. (2). Dry land. (2). Sea.

Page 385.

We will begin from the conclusion/output of the approximate boundary condition for normal to the surface of the section of the component of electric field in the first medium, which can be done completely directly, passing boundary conditions (1.01) for the tangential components of E and H. This gives the possibility to formulate integral equation directly for normal component E_z.

Precise boundary conditions for normal components $E_z^{(1)}$ and $E_z^{(2)}$ electric vector on the flat surface of the section of media 1 and 2 can be registered in the form

$$\left.\begin{aligned} (\varepsilon_1\omega + i4\pi\sigma_1) E_z^{(1)} &= \\ = (\varepsilon_2\omega + i4\pi\sigma_2) E_z^{(2)}&, \\ \frac{\partial E_z^{(1)}}{\partial z} = \frac{\partial E_z^{(2)}}{\partial z}&. \end{aligned}\right\} \quad (1.02)$$

In this case it is accepted that the dependence of fields on the time is given by factor $e^{-i\omega t}$.

The first of these equations expresses, as usual, the continuity of normal component of full current with the passage through the surface, whereas the second directly is obtained from that condition that in both media div E=0 and that on the very interface $\frac{\partial E_x^{(1)}}{\partial x} = \frac{\partial E_x^{(2)}}{\partial x}$; $\frac{\partial E_y^{(1)}}{\partial y} = \frac{\partial E_y^{(2)}}{\partial y}$ due to the continuity of the tangential components of electric field.

If the second medium possesses considerably larger conductivity, than the first, then it is possible to assume approximately (Fig. 2)

$$E_z^{(2)} = (E_z^{(2)})_{z=0}\, e^{-ik_2 z},$$
$$k_2^2 = \frac{(\varepsilon_2\omega + i4\pi\sigma_2)\,\mu_2\omega}{c^2} \equiv \frac{\eta_2\mu_2\omega^2}{c^2}, \quad \mathrm{Im}\,(k_2) > 0, \qquad (1.03)$$

moreover it is assumed that $\left(\frac{\omega}{c\,|k_2|}\right)^2 \ll 1$.

With the substitution in equations (1.02) this leads, after the elimination of unknown value $(E_z^{(2)})_{z=0}$, to the relationship/ratio

$$\left(\frac{\partial E_z^{(1)}}{\partial z}\right)_{z=0} = -2\pi\alpha\,(E_z^{(1)})_{z=0},$$
$$\alpha = \frac{i\mu_2\omega\,(\varepsilon_1\omega + i4\pi\sigma_1)}{2\pi k_2 c^2} = \frac{i\mu_2 k_1^2}{2\pi\mu_1 k_2}. \qquad (1.04)$$

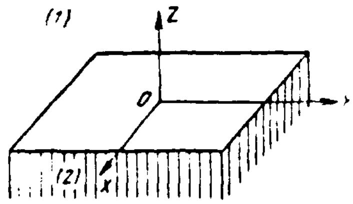

Fig. 2. Location of coordinate axes on the interface of two media.

Page 386.

This relationship/ratio in our case is fulfilled, of course, only approximately; however in the majority of the cases interesting us it is made with the sufficiently high accuracy (except dry soils). We will take it therefore for the boundary condition for E_z on the entire surface of dry land, disregarding the error, which ensues from those sections of the dry lands which directly border on sea, since width of corresponding band will be in the cases interesting us, generally speaking, it is small in comparison with the length of the wave incident to the coast (see further paragraph 2).

With the help of this boundary condition it is possible to formulate integral equation for E_z; its solution gives complete solution of the problem interesting us in that approximation/approach such as corresponds to the assumptions, done during his conclusion/output.

2. Formulation of the problem and the selection of fundamental integral equation.

IMET -[Institute of Metallurgy im. A. A. Baykov] in the form to examine a question about the incidence of electromagnetic waves from sea to the dry land or conversely, let us form the basis of of examination the following idealization of task: it is necessary to study the propagation of electromagnetic waves in the upper half-space (air), in which are arranged/located some assigned emitters, moreover on surface of z=0 has region f (dry land [1], see Fig. 3), where normal to the surface component of electric field satisfies condition (1.04), i.e.

$$\left(\frac{\partial E_z}{\partial z}\right)_{z=0} = -2\pi\alpha\,(E_z)_{z=0}.$$

FOOTNOTE [1]. It is not compulsory singly connected; for example, several islands. ENDFOOTNOTE.

On the remaining part of plane z=0 we consider as that carried out condition $\left(\frac{\partial E_z}{\partial z}\right)_{z=0} = 0$ (sea, considered as ideal conductor), which is obtained from condition (1.04) with $\sigma_2=\infty$.

This idealization of task, obviously, corresponds to the case

when the surface of dry land is flat/plane, horizontal and it is put out from the water only insignificantly in comparison with the wavelength. Small further error is introduced because we is permitted the suitability of boundary condition (1.04) on the entire surface of dry land, i.e., even at the points, close to the shore line where they, generally speaking, cease to be accurate.

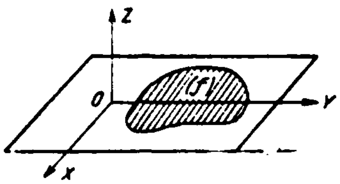

Fig. 3. Sections of interface with different boundary conditions.

Page 387.

We pass to the conclusion/output of the fundamental integral equation of task, which we will write for normal to the surface component E_z of electric field.

We proceed from the fact that E_z satisfies in air equation [1]

$$\Delta E_z + k^2 E_z = -4\pi f_z, \qquad (2.01)$$

where

$$k^2 \equiv k_1^2 = \frac{(\varepsilon_1\omega + i4\pi\sigma_1)\,\mu_1\omega}{c^2},$$

and f_z is certain known function from the coordinates, wholly determined by the assignment of primary (exciting) currents.

FOOTNOTE [1]. Beginning from this place, mark "1" in the designation of wave number and field in air will lower. ENDFOOTNOTE.

The case of the absence of absorption in air ($\sigma_1=0$) we will consider

as the limiting case of very small absorption ($\sigma_1 \to +0$).

Let us consider the range, indicated in Fig. 4, limited by plane AB of the section of media 1 and 2, by the hemisphere of radius $R \to \infty$ and infinitesimal sphere of a radius ρ, circumscribed around the arbitrary point M. Applying to this region Green's formula

$$\int_{(V)} (\psi \Delta\varphi - \varphi \Delta\psi)\, dV = \int_{(S)} \left(\psi \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial\psi}{\partial n}\right) dS,$$

when $\psi = \frac{e^{ikr}}{r}$, $\varphi = E_z$ and $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ we will obtain the following relationship/ratio:

$$4\pi\varphi_M = 4\pi E_z(x, y, z) = 4\pi E_z^0(x, y, z) + \\ + \int_{(AB)} \left[\left(\frac{\partial E_z}{\partial n}\right)_0 \frac{e^{ikr}}{r} - (E_z)_0 \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right] dS. \qquad (2.02)$$

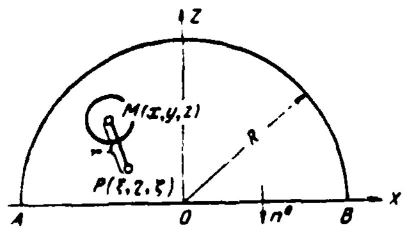

Fig. 4. Range of integration in Green's formula.

Page 388.

Here $E_z^0 = \int f_z \frac{e^{ikr}}{r} dV$ is the "primary" field which would create in the space the primary currents, if there is no interface AB, whereas integral term expresses the influence of interface.

Noting, that

$$\left(\frac{\partial E_z}{\partial n}\right)_0 = -\left(\frac{\partial E_z}{\partial z}\right)_0$$

and that $\frac{\partial E_z}{\partial z}$ is equal to zero on the ideally conducting parts of surface AB, whereas on the surface (f) there will be $\left(\frac{\partial E_z}{\partial z}\right)_0 = -2\pi\alpha\,(E_z)_0$, we will obtain from (2.02).

$$4\pi\,(E_z)_M = 4\pi\,(E_z^0)_M + 2\pi\alpha \int_{(f)} (E_z)_0 \frac{e^{ikr}}{r} df -$$

$$- \int_{AB} (E_z)_0 \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right) dS_{AB}. \qquad (2.03)$$

Applying this relationship/ratio to point M, which lies on surface itself (f), and noting that in this case it will be

$$-\int (E_z)_0 \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right) dS_{AB} = 2\pi\,(E_z)_M.$$

we obtain finally, substituting mark M in E_z by sign 0,

$$(E_z)_0 = 2\,(E_z^0)_0 + \alpha \int_{(f)} (E_z)_0 \frac{e^{ikr}}{r}\, df. \qquad (2.04)$$

This is a fundamental integral equation of our task.

Let us note that if entire/all plane AB was ideally conducting, then equation (2.04) would give simply

$$E_z = 2\,(E_z^0)_0, \qquad (2.05)$$

i.e. the duplication of actual value $(E_z)_0$ on the interface in comparison with value $(E_z^0)_0$ of "primary" field on it. This, obviously, is expressed usual law of reflection for the ideally conducting plane.

After finding from equation (2.04) value $(E_z)_0$ and substituting it in the right side of formula (2.03), we will be able to obtain value E_z at any point M above the plane, i.e., we will have the complete solution of stated problem.

Page 389.

3. Case of the unlimited rectilinear shore.

We pass to the case when surface f of dry land - half-plane. We

choose the coordinate axes as shown in Fig. 5: Y axis is directed along the coast, and axis X - to the side dry land.

Taking into account, that any applied field $(E_z^0)_0$ can be decomposed into Fourier's integral (series/row) along the axis Y, it is possible to be bounded to the case purely sinusoidal along the axis Y of field, i.e., to suppose that

$$(E_z^0)_0 = F^0(x)\,e^{isy}, \qquad (3.01)$$

where s can have any real value.

Assuming/setting in this case in formula (2.04)

$$(E_z)_0 = F(x)\,e^{isy}, \qquad (3.02)$$

we we can register it now thus:

$$F(x) = 2F^0(x) + \alpha \int_0^{\infty} F(\xi)\,d\xi \int_{-\infty}^{+\infty} \frac{e^{i\left(s\eta + k\sqrt{(x-\xi)^2+\eta^2}\right)}}{\sqrt{(x-\xi)^2+\eta^2}}\,d\eta. \qquad (3.03)$$

It is not difficult to show that the integral entering here on η has a value

$$\int_{-\infty}^{+\infty} \frac{e^{i\left(s\eta + k\sqrt{(x-\xi)^2+\eta^2}\right)}}{\sqrt{(x-\xi)^2+\eta^2}}\,d\eta = i\pi H_0^{(1)}\left(\sqrt{k^2-s^2}\,|x-\xi|\right), \qquad (3.04)$$

where $H_0^{(1)}$ - Hankel function of the first zero-order order.

formula (3.03) gives therefore, if we assume $m = \sqrt{k^2 - s^2}$,

$$F(x) = 2F^0(x) + i\pi\alpha \int_0^{\infty} F(\xi)\,H_0^{(1)}(m\,|x-\xi|)\,d\xi. \qquad (3.05)$$

This is the equation to solution of which is reduced with the

done simplifications the solution of the problem of coastal refraction in the case in question.

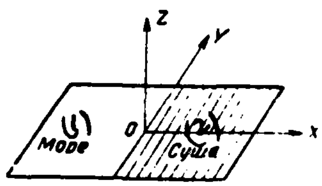

Fig. 5. Location of coordinate axes in the case of the unlimited rectilinear shore.

Key: (1). Sea. (2). Dry land.

4. General/common/total theorem about the solution of integral equation.

The integral equation to which is given stated problem, is a special case of the equation of the form

$$f(x) = g(x) + \int_0^\infty k(|x-\xi|)\, f(\xi)\, d\xi. \qquad (4.01)$$

Page 390.

General/common/total research of equation (4.01) is given in Foch's work [19] (addition of this 1 book). We will give here only those results of this research which are necessary for solving equation (3.05).

Kernel k(x) (which we we assume even) is such that not only it

auto, but also function $k_1(x) = e^{c|x|}k(x)$ will be for certain c>0 the absolutely integrated function with the bounded variation in the infinite gap/interval.

In the case of kernel $H_0^{(1)}(m|x-\xi|)$ this condition will be satisfied, if we consider that is certain, at least minimum, the conductivity of air; in this case it is possible to take as by the constant c any number, which satisfies the inequality

$$0 < c < \operatorname{Im}(m).$$

For solving equation (4.01) we construct the function

$$K(w) = \int_{-\infty}^{+\infty} e^{iwx} k(x)\, dx \qquad (4.02)$$

and we investigate, it is converted into one at the real values of w. For kernel $H_0^{(1)}(m|x-\xi|)$ it into one is not converted, so that we have here a matter concerning the simplest of the cases, dismantled/selected in [19]. Further, we compose the integral

$$\chi_1(w) = -\frac{1}{2\pi i} \int_{-ic-\infty}^{-ic+\infty} \frac{\lg[1-K(u)]}{u-w}\, du \qquad (4.03)$$

and we construct the function

$$\psi_1(w) = e^{\chi_1(w)}. \qquad (4.04)$$

As it is proved in [19], this function is holomorphic and does not have zero in the upper half-plane w (or in the band -c<Im(w)<c) and satisfies the functional equation

$$\psi_1(w)\,\psi_1(-w)\,[1-K(w)] = 1. \qquad (4.05)$$

Let us turn now to the absolute term g(x) of the integral equation proposed. Let g(x) be the function, absolutely integrated

also with the bounded variation in the infinite gap/interval.

Page 391.

We can compose then the function

$$G_1(w) = \int_0^\infty e^{ixw} g(x)\, dx \tag{4.06}$$

and express through it and through $\psi_1(w)$ the unknown solution of integral equation. Namely, we will have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixw} F_1(w)\, dw, \tag{4.07}$$

where $F_1(w)$ is determined from the equality and

$$F_1(w) = \frac{\psi_1(w)}{2\pi i} \int_{-\infty}^{+\infty} \frac{\psi_1(-u)\, G_1(u)}{u - w}\, du \tag{4.08}$$

with Im(w)>0 and from the equality

$$F_1(w) = \psi_1(w)\, \psi_1(-w)\, G_1(w) + \frac{\psi_1(w)}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\psi_1(-u)\, G_1(u)}{u - w}\, du \tag{4.09}$$

with Im(w)<c. In the band 0<Im(w)<c both expressions coincide.

In the proof of the fact that expression (4.07) actually/really satisfies integral equation (4.01) proposed, is used the fact that function $F_1(w)$ is holomorphic in the upper half-plane, whereas the function

$$\Phi(w) = F_1(w) - K(w)\, F_1(w) - G_1(w)$$

is holomorphic in the lower half-plane, moreover both functions $F_1(w)$ and $\Phi(w)$ become zero with $|w| \to \infty$.

Proof itself we do not here give.

The set of the formulas of this paragraph composes the theorem about the solution of the integral equation of form (4.01).

These formulas considerably are simplified in that special case when the absolute term of integral equation is exponential function. Actually/really, let us assume

$$g(x) = e^{ipx} \quad (\mathrm{Im}\,(p) > 0). \tag{4.10}$$

According to formula (4.06) we will then have

$$G_1(w) = \frac{i}{w+p}. \tag{4.11}$$

Integral (4.08) is reduced to the deduction at point w=-p and gives

$$F_1(w) = i\,\frac{\psi_1(w)\,\psi_1(p)}{w+p} \tag{4.12}$$

- the expression, symmetrical relative to w and p. The solution of equation takes the form

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{-ixw}\,\frac{\psi_1(w)\,\psi_1(p)}{w+p}\,dw. \tag{4.13}$$

Page 392.

It is necessary to remember that at point x=0 function f(x), determined by integral (4.07), has a gap, moreover f(-0)=0 and f(+0)=2f(0,.

In the case (4.13) we have

$$f(+0) = \psi_1(p). \tag{4.14}$$

5. Solution of the integral equation of coastal refraction.

We can now use our general formulas to the solution of the integral equation of the coastal refraction:

$$F(x) = 2F^0(x) + i\pi\alpha \int_0^\infty H_0^{(1)}(m|x-\xi|)\, F(\xi)\, d\xi. \qquad (5.01)$$

In this case we will be bounded to the case of attack to the coast of plane wave, in accordance with how let us place

$$F^0(x) = e^{imx}, \qquad (5.02)$$

considering it thereby the wave amplitude on the border of dry land and sea equal to one. In the absence of absorption in air we can assume $k=2\pi/\lambda$, where λ - wavelength in air. Then it will be

$$m = \frac{2\pi}{\lambda}\cos\vartheta; \quad s = \frac{2\pi}{\lambda}\sin\vartheta, \qquad (5.03)$$

where ϑ is an angle between the wave front and the shore line.

We will; however, (as, until now,), consider the case of absence absorbed as the limiting case of very small absorption and to the transition to the limit we will count the alleged part of m of positive. The determined by equations (5.01) and (5.02) function F(x) represents analytic function from x which can be examined not only for the real ones, but also for the complex values x. If we will place

$$-imx = x_1, \qquad (5.04)$$

that we we can determine first this function for the the real x_1 (i.e., for the complex values of coordinate x), and then already to switch over to such (complex) values of x_1, to which correspond the real values of coordinates x. This examination is convenient, since it makes it possible to more easily isolate the branches of ambiguous functions interesting us.

Page 393.

Let us assume

$$F(x) = 2f(x_1), \quad \frac{2\pi i\alpha}{m} = a \tag{5.05}$$

and let us compose equation for $f(x_1)$ on the assumption that x_1 really and varies from 0 to $+\infty$. We will obtain

$$f(x_1) = e^{-x_1} + \frac{ia}{2}\int_0^\infty H_0^{(1)}(i|x_1-\xi|)\, f(\xi)\, d\xi \tag{5.06}$$

or

$$f(x_1) = e^{-x_1} + \frac{a}{\pi}\int_0^\infty K_0(|x_1-\xi|)\, f(\xi)\, d\xi, \tag{5.07}$$

if we is expressed the Hankel function $H_0^{(1)}$ through MacDonald's function K_0.

Equation (5.07) we will solve with the help of the theorem, formulated in the previous paragraph. Converting kernel according to (4.02) according to Fourier's formula, we will obtain

$$K(w) = \frac{a}{\pi}\int_{-\infty}^{+\infty} e^{-iwx} K_0(|x|)\, dx = \frac{a}{\sqrt{w^2+1}}. \tag{5.08}$$

We compute by formula (4.03) function $\chi_1(w)$. Assuming Im(w)>0, we can write

$$\chi_1(w) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \lg\left(1-\frac{a}{\sqrt{u^2+1}}\right)\frac{du}{u-w}. \qquad (5.09)$$

Derivative of this integral on w is expressed in the final form. Assuming/setting

$$b=\sqrt{1-a^2}, \quad \mathrm{Re}(b)>0, \qquad (5.10)$$

we we will obtain after the elementary calculations

$$\frac{d\chi_1}{dw} = \frac{1}{2(w+i)} - \frac{1}{2(w+ib)} +$$
$$+\frac{1}{2\pi(w^2+b^2)}\left\{b\lg\frac{b+ia}{b-ia} - ia\frac{w}{\sqrt{1+w^2}}\lg\frac{w+\sqrt{1+w^2}}{w-\sqrt{1+w^2}}\right\}. \qquad (5.11)$$

This expression is holomorphic in the upper half-plane w; in particular, it is easy to check that point w=ib is not pole for $\chi_1'(w)$.

Page 394.

In order to simplify the obtained expression, let us assume

$$b=\cos\sigma, \quad a=\sin\sigma, \quad \left(-\frac{\pi}{2}<\mathrm{Re}(\sigma)<\frac{\pi}{2}\right), \qquad (5.12)$$
$$w=i\cos\tau, \quad \sqrt{1+w^2}=\sin\tau,$$

moreover Re(sin τ)>0, and let us compose derivative of χ_1 on τ. We will have

$$\frac{d\chi_1}{d\tau} = -\frac{\sin\tau}{2(\cos\tau+1)} + \frac{\sin\tau}{2(\cos\tau+\cos\sigma)} + \\ + \frac{\tau+\sigma}{2\pi\sin(\tau+\sigma)} - \frac{\tau-\sigma}{2\pi\sin(\tau-\sigma)}. \quad (5.13)$$

Hence

$$\chi_1 = \frac{1}{2}\lg\frac{\cos\tau+1}{\cos\tau+\cos\sigma} + \frac{1}{2\pi}\int\limits_{\tau-\sigma}^{\tau+\sigma}\frac{u}{\sin u}\,du, \quad (5.14)$$

since with $\tau=i\infty$ must be $\chi_1 = 0$.

Thus, for function $\psi_1(w) = \psi_1(i\cos\tau)$ is obtained the expression

$$\psi_1(i\cos\tau) = \sqrt{\frac{\cos\tau+1}{\cos\tau+\cos\sigma}}\exp\left(\frac{1}{2\pi}\int\limits_{\tau-\sigma}^{\tau+\sigma}\frac{u}{\sin u}\,du\right). \quad (5.15)$$

It is not difficult to check that

$$\psi_1(i\cos\tau)\,\psi_1(i\cos(\pi-\tau)) = \frac{\sin\tau}{\sin\tau-\sin\sigma}. \quad (5.16)$$

Since in the variable/alternating σ, τ kernel K(w) takes the form

$$K(w) = \frac{\sin\sigma}{\sin\tau}, \quad (5.17)$$

the relationship/ratio (5.16) shows that function $\psi_1(w)$ it actually/really satisfies functional equation (4.05).

Substituting (5.15) in (4.13), we will obtain the solution of our integral equation (5.07). For the research of solution it is necessary, however, to convert the obtained integral so that it rapidly would descend. For this it is necessary to replace integration for the real axis with the equivalent duct/contour, which lies at the lower half-plane by complex variable w. In the upper

half-plane the function regarding is holomorphic and does not have zero. Singular points $\psi_1(w)$ in the lower half-plane more easily in all to find with the help of functional equation (4.05), which gives

$$\psi_1(w)=\frac{\sqrt{1+w^2}}{\psi_1(-w)\,(\sqrt{1+w^2}-\sin\sigma)}. \qquad (5.18)$$

Page 395.

Hence it is apparent that singular points they will be: branch point w=-i and pole $w=-ib=-i\cos\sigma$ (latter only when the real part of value $a=\sin\sigma$ is positive).

Reducing the duct/contour to the loop around point w=-i, which covers negative imaginary axis, we will obtain with Re(a)>0:

$$f(x_1)=\frac{\psi_1(i)}{\psi_1(i\cos\sigma)}\,\frac{i+\cos\sigma}{\cos\sigma}\,e^{-x_1\cos\sigma}-$$
$$-\frac{\sin\sigma}{2\pi}\int_{-\infty}^{+\infty}e^{-x_1\operatorname{ch}t}\,\frac{\psi_1(i)}{\psi_1(i\operatorname{ch}t)}\,\frac{\operatorname{ch}t+1}{\operatorname{ch}^2t-\cos^2\sigma}\,dt \qquad (5.19)$$
$$\overset{(1)}{[\text{при}}\ \operatorname{Re}(\sin\sigma)>0].$$

Key: (1). with.

where the first term presents deduction in pole w=-ib.

However, in the case of Re(a)<0 deduction to take is not necessary, and we will have

$$f(x_1) = -\frac{\sin\sigma}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-x_1 \operatorname{ch} t}\frac{\psi_1(i)}{\psi_1(i \operatorname{ch} t)}\frac{\operatorname{ch} t+1}{\operatorname{ch}^2 t-\cos^2\sigma}\,dt \qquad (5.20)$$

$$[\overset{(1)}{\text{при}}\ \operatorname{Re}(\sin\sigma) < 0].$$

Key: (1). with.

We found a strict solution of the integral equation proposec, besides in the form, valid not only for the positive values of x_1, but also for the complex values x_1 with the non-negative alleged part; consequently, and for x_1=-imx, where x - real coordinate.

The approximation formulas for $f(x_1)$ will be derived in the following paragraph.

6. Approximate form of the solution of integral equation.

During the research of the obtained solution of integral equation it is necessary to remember that this equation is only approximate and it is correct only with the observance of the condition

$$\frac{\omega}{\mu_2\sqrt{\varepsilon_2^2\omega^2+(4\pi\sigma_2)^2}} \ll 1. \qquad (6.01)$$

With the help of the introduced above values a and ϑ this condition can be written in the form

$$|a|^2\cos^2\vartheta \ll 1. \qquad (6.02)$$

Page 396.

At not too low values of cos ϑ (with not too oblique incidence in the wave) this condition is equivalent to the requirement of the smallness of value itself $|a|$ and, consequently, also entering our formulas of formula parameter σ. However, at the low values $|\sigma|$ our formulas can be simplified. But this simplification must be made so that the formulas would remain valid not only for the final ones, but also for how conveniently high values of x_1 (for the large distances from the coast into the depth of dry land; let us recall that x_1=-ikx cos ϑ). Having this in mind, we will compute the integral, which stands in formulas (5.19) and (5.20), first for the high values of x_1 without assuming σ small, and then let us switch over to the case of small ones σ.

The main section of integration will be the vicinity of point t=0; near this point we can the slowly varying factors

$$\frac{\psi_1(i)}{\psi_1(i\,\mathrm{ch}\,t)} \text{ и } \frac{\mathrm{ch}\,t+1}{\mathrm{ch}\,t+\cos\sigma}$$

replace with their values with t=0. These values are equal with respect to unity and value $\sec^2 \sigma/2$. After doing these neglects, we will obtain

$$\left.\begin{array}{c} f(x_1) = Ae^{-x_1 \cos\sigma} - \frac{1}{\pi}\,\mathrm{tg}\,\frac{\sigma}{2}\, I(x_1, \sigma) \\ [\text{при (1)}\ \mathrm{Re}\,(\sin\sigma) > 0], \\ f(x_1) = -\frac{1}{\pi}\,\mathrm{tg}\,\frac{\sigma}{2}\, I(x_1, \sigma) \\ [\text{при (1)}\ \mathrm{Re}\,(\sin\sigma) < 0], \end{array}\right\} \tag{6.03}$$

Key: (1). with.

where we assumed

$$A = \frac{\psi_1(l)}{\psi_1(i\cos\sigma)}\,\frac{1+\cos\sigma}{\cos\sigma}; \tag{6.04}$$

$$I(x_1, \sigma) = \int_{-\infty}^{+\infty} e^{-x_1 \operatorname{ch} t}\,\frac{dt}{\operatorname{ch} t - \cos\sigma}. \tag{6.05}$$

Latter/last integral is equal to

$$I(x_1, \sigma) = 2\int_0^{\infty} e^{u\cos\sigma} K_0(x_1 + u)\,du, \tag{6.06}$$

as of this easy to be convinced with the help of the differential equation

$$\frac{dI}{dx_1} + \cos\sigma I = -2K_0(x_1). \tag{6.07}$$

Page 397.

Since we assume x_1 large, then MacDonald's function under the integral we can replace with its asymptotic expression, after which we will obtain

$$I(x_1\ \sigma) = \frac{\pi}{\sin\frac{\sigma}{2}}\, e^{-x_1\cos\sigma}\left[\pm 1 - \Theta\left(\sqrt{2x_1}\,\sin\frac{\sigma}{2}\right)\right], \tag{6.08}$$

where $\theta(\xi)$ is determined by the equality

$$\theta(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\xi^2} d\xi. \qquad (6.09)$$

but sign with unity in (6.08) the same as the sign of real part sin $\sigma/2$ or sin σ. [According to the condition (5.12) we have $-\pi/2<\mathrm{Re}(\sigma)<\pi/2$, in consequence of which the real parts of values σ, sin $\sigma/2$, sin σ have one and the same sign].

Assuming/setting

$$\xi = \sqrt{2x_1} \sin \frac{\sigma}{2} \qquad (6.10)$$

and substituting (6.08) in (6.03), we will have with Re(sin σ)>0

$$f(x_1) = e^{-x_1 \cos \sigma}\left[A - \sec \frac{\sigma}{2} + \sec \frac{\sigma}{2}\, \theta(\xi)\right] \qquad (6.11)$$

and with Re(sin σ)<0

$$f(x_1) = e^{-x_1 \cos \sigma}\left[\sec \frac{\sigma}{2} + \sec \frac{\sigma}{2}\, \theta(\xi)\right]. \qquad (6.12)$$

As it is indicated in the beginning of this paragraph, is most interesting the case of small ones σ. In this case both expressions (6.11) and (6.12) coincide and can be written in the form

$$f(x_1) = e^{-x_1} \varphi(\xi), \qquad (6.13)$$

where

$$\varphi(\xi) = e^{\xi^2}[1 + \theta(\xi)], \qquad (6.14)$$

moreover value ξ can be placed equal to

$$\xi = \sqrt{\frac{x_1}{2}} \sin \sigma = a \sqrt{\frac{x_1}{2}}. \qquad (6.15)$$

But if σ is small, then formulas (6.13) and (6.14) are valid with what conveniently (but not only at the high) values of x_1.

Page 398.

In fact, if with small σ value x_1 will be finite (not large), then value ξ will be small $\varphi(\xi)$ will differ little from one, but f(x_1) will be approximately equally to e^{-x_1}, to absolute term in integral equation (5.07), as must be. Thus, if σ is small, then formulas (6.13) and (6.14), of the derived under the assumption large x_1, are suited also without this assumption.

To us remains to switch over from variable/alternating x_1 to coordinate x and to substitute the obtained solution in the expression for the amplitude of field. In this case is convenient for the case of small ones σ to introduce value

$$\rho = \frac{ik^3 x}{2k_2^2 \cos\vartheta}, \qquad (6.16)$$

analogous to Sommerfeld numerical distance. If it is possible to disregard bias currents in the earth/ground in comparison with the conduction current, then value ρ will be real. In that case ρ is a distance from the coast, expressed on the known scale and counted along the ray/beam (from the point of intersection of its with the coast).

Substituting in (6.13) the values

$$a = -\frac{k}{k_1 \cos\vartheta}, \quad x_1 = -ikx\cos\vartheta, \qquad (6.17)$$

we will obtain for our previous variable/alternating ξ the expression

$$\xi = i\sqrt{\rho}. \qquad (6.18)$$

With this value ξ the "attenuation factor" φ can be written in the form

$$\varphi = e^{-\rho}\left[1 + i\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\rho}} e^{t^2}\,dt\right]. \qquad (6.19)$$

In the case examined incident wave according to (3.01), (5.02) and (5.03) took the form

$$(E_z^0)_n = e^{ik(x\cos\theta + y\sin\theta)}. \qquad (6.20)$$

Our formulas show then that the complete field above the dry land can be represented in the form

$$(E_z)_n = 2\varphi e^{ik(x\cos\theta + y\sin\theta)}, \qquad (6.21)$$

where φ has value (6.19).

Page 399.

Mathematical additions.

Page 400. No Typing.

Page 401.

Addition 1.

Some integral equations of mathematical physics [1].

FOOTNOTE [1]. Foch, 19744. ENDFOOTNOTE.

In addition 1 is given the complete mathematical theory of linear integral second-order equations (i.e. with the unknown function, which stands both under the integral sign and outside it) with the semi-infinite limits and the kernel, which depends on the absolute value of a difference in the arguments. Such equations are encountered in many problems of mathematical physics, for example in the problem about the coastal refraction (Chapter 18), about the radiation/emission of semi-infinite waveguide with the open end/lead, about the absorption and the scattering of light in the atmosphere and in other tasks.

Are formulated the conditions, which must be superimposed to the kernel of equation and to the assigned function so that there would be the unique solution with the assigned properties (limitedness, tendency toward zero to infinity, representability in the form of the sum of function with the bounded variation in the infinite gap/interval and the continuous function). With the help of the kernel, converted according to Fourier, is comprised the characteristic equation of task. Essential role plays the isolation/liberation in it of the factors, which present complex variable function with the specific properties in the upper and lower half-plane (method of factorization).

If characteristic equation does not have real roots, then for the existence of the solution with the properties indicated it is sufficient so that the assigned function would be absolutely integrated and would have the bounded variation in the infinite gap/interval. If there is 21 real roots, then the assigned function must satisfy 1 to the conditions of the orthogonality (and, furthermore, somewhat more rapidly to decrease at infinity). In the case of the absence of real roots the corresponding homogeneous equation does not have the limited (and even it is not too rapid increasing) solutions. In the case 21 of real roots it has exactly 1

of the limited solutions; these solutions they enter into the conditions of orthogonality.

Research is conducted with the wide use of properties of Fourier integrals in the complex plane, and the solution is obtained in the form of contour integrals, i.e., in the explicit form, which allows/assumes numerical calculations.

Page 402.

Introduction.

In the series of problems of mathematical physics are encountered the integral equations of the form

$$f(x) = g(x) + \int_0^{\infty} k(x-y) f(y)\, dy \tag{0.1}$$

with the symmetrical kernel.

$$k(x-y) = k(|x-y|), \tag{0.2}$$

depending on the absolute value of a difference in two arguments.

Solution of integral equation with kernel (0.2) and with the variable/alternating upper limit of x (equation of Volterra) is given in our previous work (1922) [34]. It is examined also in book Doetsh [43], dedicated to the Laplace transform.

However, equation with the infinite upper limit is not still completely solved. In the literature there is only a research of the corresponding homogeneous equation (also in the more general case when the condition of symmetry (0.2) is not satisfied) [35]. As far as nonhomogeneous equation is concerned, it, apparently, is not investigated.

The task about the nonhomogeneous equation has, however, independent interest. In fact, in mathematical physics are important those cases when the solution of problem only, but exactly these cases during the study of uniform task usually are not examined, since with satisfaction of the conditions, which escape/ensue from the physical requirements, the latter does not have another solution, except zero.

The purpose of this work is the research and the solution both of the nonhomogeneous and homogeneous equation of form (0.1).

1. Conversion of equation.

In the equation

$$f(x)=g(x)+\int_0^\infty k(x-y)f(y)\,dy \qquad (1.01)$$

function g(x) is assigned only for x⩾0. The unknown solution f(x) also is required to determine only for x⩾0. But we can agree to consider that

$$f(x) = 0 \text{ при}^{(1)} \; x < 0. \qquad (1.02)$$

Key: (1). with.

So that equation (1.01) would occur also for x<0, we will subordinate function g(x) to the condition

$$g(x) = -\int_0^{\infty} k(x-y) f(y)\,dy \text{ при}^{(1)} \; x < 0. \qquad (1.03)$$

Key: (1). with.

Page 403.

This permits for us to represent equation (1.01) in the form

$$f(x) = g(x) + \int_{-\infty}^{+\infty} k(x-y) f(y)\,dy \qquad (1.04)$$

and to consider the entering it functions as for the positive ones, so also for the negative values of arguments.

The solution of equation (1.04) is obtained very simply with the help of the Fourier transform. In fact, let us assume that functions

k(x), g(x) and f(x) are such, that to them is applicable Fourier's formula [1].

FOOTNOTE [1]. Of course the applicability of Fourier's formula to f(x) must be still proved, moreover in the proof it is necessary to proceed from one or the other conditions, assigned on k(x) and on g(x). ENDFOOTNOTE.

Let us introduce the designations

$$K(w) = \int_{-\infty}^{+\infty} e^{iwx} k(x)\, dx, \qquad (1.05)$$

$$F(w) = \int_{-\infty}^{+\infty} e^{iwx} f(x)\, dx, \qquad (1.06)$$

$$G(w) = \int_{-\infty}^{+\infty} e^{iwx} g(x)\, dx. \qquad (1.07)$$

Multiplying (1.04) on $e^{iwx}\,dx$ and integrating by x in the limits from $-\infty$ to $+\infty$, we come to the relationship/ratio

$$F(w) = G(w) + K(w)\, F(w), \qquad (1.08)$$

whence F(w) is obtained by algebraic path. The substitution of this value F(w) into the formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iwx} F(w)\, dw \qquad (1.09)$$

and gives solution of equation (1.04).

Page 404.

Let us note that for the equation with kernel (0.2) and with the variable/alternating upper limit of x function F(w) in (1.09) is defined, as shown in [34], from the relationship/ratio

$$F(w) = G_1(w) + K_1(w)F(w), \qquad (1.10)$$

where $G_1(w)$ - the integral

$$G_1(w) = \int_0^\infty e^{iwx} g(x)\,dx, \qquad (1.11)$$

but $K_1(w)$ is connected with K(w) just as $G_1(x)$ with G(w). Relationship/ratio (1.10), just as (1.08), purely algebraic.

In the case of integral equation (1.01) the matter is complicated by the fact that function G(w) in (1.08) to us is unknown.

Counting g(x) by directly assigned only for positive x, we can assume/set by known integral (1.11), but not integral (1.07). Concerning (1.07), as a result of (1.02) and (1.03)

$$G(w) = G_1(w) - \int_{-\infty}^{0} e^{iwx}\,dx \int_{-\infty}^{+\infty} k(x-y)f(y)\,dy. \qquad (1.12)$$

Integral on y is equal to

$$\int_{-\infty}^{+\infty} k(x-y)f(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux} K(u)F(u)\,du. \qquad (1.13)$$

Integral on x we write in the form of a difference in the integrals from -∞ to +∞ and from 0 to ∞; the first of them we compute according to Fourier's formula and we transfer into left side (1.12). We obtain

$$\mathcal{G}(w) + K(w)F(w) = G_1(w) + \frac{1}{2\pi}\int_0^\infty e^{iwx}\,dx \int_{-\infty}^{+\infty} e^{-iux} K(u)F(u)\,du. \tag{1.14}$$

Thus, relationship/ratio (1.08) takes the form

$$F(w) = G_1(w) + \frac{1}{2\pi}\int_0^\infty e^{iwx}\,dx \int_{-\infty}^{+\infty} e^{-iux} K(u)F(u)\,du. \quad (1.15)$$

If we have in mind formula (1.13), then equation (1.15) can be obtained directly from (1.01) or (1.04) after multiplication by $e^{iwx}\,dx$ and the integrations for x from 0 to ∞.

We let us assume that function g(x) is absolutely integrated and has the bounded variation in entire infinite gap/interval. Then the converted function $G_1(w)$ will be, regarding (1.11), it is holomorphic in the upper half-plane, it is continuous up to the real axis, and it will decrease both in the upper half-plane and on the real axis conversely $|w|$.

Page 405.

Relative to kernel k(x) we let us assume that there is this positive constant c, that not only kernel itself, but also the function

$$k_1(x) = e^{c|x|}k(x) \quad (c > 0) \qquad (1.16)$$

is absolutely integrated and has the bounded variation in the infinite gap/interval. Hence, in particular, it follows that with all x

$$|k(x)| < Me^{-c|x|}, \tag{1.17}$$

where M - certain positive constant.

Further according to (0.2) kernel k(x) must be even function from x. Under these conditions the converted kernel K(w) will prove to be holomorphic within the band

$$-c < \operatorname{Im}(w) < +c \tag{1.18}$$

and it will in it decrease conversely $|w|$. Furthermore, it will be even function from w.

The properties of function F(w) will be in detail investigated below. In any case, from condition (1.02) it follows that integral (1.06) coincides with the integral

$$F_1(w) = \int_0^\infty e^{iwx} f(x)\,dx. \tag{1.19}$$

and that, therefore, F(w) must be the function, holomorphic in the upper half-plane. We will show also that F(w) will decrease in the upper half-plane conversely $|w|$.

As a result of the holomorphy of functions F(w) and $G_1(w)$ with Im(w)>0 we can examine equation (1.15) for the complex values of w with the positive imaginary part. For such w we can first fulfill integration for x, after which we will obtain

$$F(w)=G_1(w)+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{K(u)F(u)}{u-w}du. \qquad (1.20)$$

In this new integral equation the terms out of the integral can be in turn, represented in the form of the integrals

$$F(w)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(u)}{u-w}du, \qquad (1.21)$$

$$G_1(w)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{G_1(u)}{u-w}du. \qquad (1.22)$$

In order to be convinced of this, it suffices to use Cauchy formula to the region, limited by real axis and another semicircumference of the unlimitedly increasing radius, which lies at the upper half-plane (in this region both functions F(w) and $G_1(w)$ are holomorphic).

Page 406.

Substituting (1.21) and (1.22) in (1.20) and transferring all members into the left side, we will obtain the equation

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Phi(u)\,du}{u-w}=0 \overset{(1)}{\text{при}}\ \operatorname{Im}(w)>0. \qquad (1.23)$$

Key: (1). with.

where we assumed

$$F(u)[1-K(u)]-G_1(u)=\Phi(u). \qquad (1.24)$$

2. Lemmas.

Subsequently for us it is necessary to repeatedly use the following lemmas:

Lemma I. Let function H(w) be holomorphic within the band

$$-a\leqslant \operatorname{Im}(w)\leqslant b, \qquad (2.01)$$

it is continuous up to its boundary it becomes zero with Re(w) →+-∞ within the band at least conversely $|w|^{\sigma}$, where $\sigma>0$.

Then it it is possible to present in the form of the sum of two functions

$$H(w)=H_1(w)+H_2(w), \qquad (2.02)$$

of which the first is holomorphic not only within the band, but also in the entire half-plane Im(w)>a, and the second - not only within the band, but also in the entire half-plane Im(w)<b.

Proof is based directly on the Cauchy formula. If a<Im(w)<b, then it is possible to assume

$$H_1(w)=\frac{1}{2\pi i}\int\limits_{ia-\infty}^{ia+\infty}\frac{H(u)}{u-w}\,du, \qquad (2.03)$$

$$H_2(w)=-\frac{1}{2\pi i}\int\limits_{ib-\infty}^{ib+\infty}\frac{H(u)}{u-w}\,du. \qquad (2.04)$$

Applying Cauchy formula to the region, limited by the straight lines Im(w)=a and Im(w)=b, we will obtain for the sum of expressions (2.03) and (2.04) value H(w). On the other hand, it is obvious that function (2.03) is holomorphic with Im(w)>a, then function (2.04) is holomorphic with Im(w)<b. Thus, the required resolution is proved.

Estimation of functions $H_1(w)$ and $H_2(w)$ at infinity is given in lemmas II and III.

Page 407.

Lemma II. If for the sufficiently large $|u|$ within band (2.01) has place the inequality

$$|H(u)| < \frac{M}{|u|^{\sigma}} \qquad (0 < \sigma \leqslant 1). \qquad (2.05)$$

then for the sufficiently large $|w|$ function $H_1(w)$ satisfies in region Im(w)>a the inequality

$$|H_1(w)| < \frac{1}{\pi |w|^{\sigma}} \left\{ M \lg \left| \frac{w}{w^*} \right| + M' \right\} \quad (\sigma < 1) \qquad (2.06)$$

or the inequality

$$|H_1(w)| < \frac{1}{\pi |w|} \left\{ M \lg |w| + M \lg \left| \frac{w}{w^*} \right| + M'' \right\} \quad (\sigma = 1). \quad (2.07)$$

where the constants M', M", just as M, do not depend on w.

In these formulas through w* is designated the alleged part w-a:

$$w^* = \operatorname{Im}(w - a). \qquad (2.08)$$

If this value is lower than half of the bandwidth, i.e., if

$$\operatorname{Im}(w) < \frac{b+a}{2}, \qquad (2.09)$$

it is necessary to replace $|w^*|$ by b-a/2.

Analogous estimation occurs for $H_2(w)$ in region Im(w)<b.

Observation. In formula (2.06) log term remains final, if point w recedes to infinity on half line inclined toward the real axis at final angle $(w = |w|e^{i\vartheta};\ \varepsilon < \vartheta < \pi - \varepsilon;\ \varepsilon > 0)$. Under this condition $H_1(w)$ it will be order $|w|^{-\sigma}$, if σ<1. However, in the formula (2.07) of two log terms it suffices then to leave the first, so that if σ=1, then function $H_1(w)$ there will be order $|w|^{-1}\lg|w|$.

Proof. If point w lies within band (2.01), then for the proof it suffices to consider that of the integrals $H_1(w)$ and $H_2(w)$, in which contour of the integration stands further away from point w. For another integral the estimation will escape/ensue from relationship/ratio $H_1(w)+H_2(w)=H(w)$.

Page 408.

After rewriting formula (2.03) in the form

$$H_1(w) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{H(u+ia)\,du}{u+ia-w} \qquad (2.10)$$

and substituting integrand by its modulus/module, we will obtain the inequality

$$|H_1(w)| < \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|H(u+ia)|\,du}{|u+ia-w|}. \qquad (2.11)$$

In integral (2.11) it is possible to decompose the way of integration in three sections: from $-\infty$ to $-A$, from $-A$ to $+A$ and from A to $+\infty$, where A - sufficiently large number, which does not depend on w. Integral on the middle section will be, obviously, order $|w|^{-1}$. However, in the integral according to the extreme sections it is possible to use estimation (2.05) for H(u)that also brings, after elementary ones, although several complicated calculations, to the formulas of lemma II.

Lemma III. If for the sufficiently large $|u|$ within band (2.01) occurs the inequality

$$|H(u)| < \frac{M}{|u|^{1+\sigma}} \qquad (0 < \sigma < 1), \qquad (2.12)$$

then for the sufficiently large $|w|$ functions $H_1(w)$ and $H_2(w)$ can be

represented (the first - in region Im(w)>a, the second - in region Im(w)<b) in the form

$$H_1(w)=\frac{iC}{w}+\frac{1}{w}H_1^*(w), \qquad (2.13)$$

$$H_2(w)=-\frac{iC}{w}+\frac{1}{w}H_2^*(w), \qquad (2.14)$$

where C - constant, and $H_1^*(w)$ and $H_2^*(w)$ satisfy the same inequalities as $H_1(w)$ and $H_2(w)$ in lemma II.

Proof. Let us introduce into formula (2.03) the identity

$$\frac{1}{u-w}=-\frac{1}{w}+\frac{u}{w(u-w)}. \qquad (2.15)$$

We will obtain for $H_1(w)$ expression (2.13), in which the constant C is equal to

$$C=\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty}H(u)\,du=\frac{1}{2\pi}\int_{ib-\infty}^{ib+\infty}H(u)\,du, \qquad (2.16)$$

and function $H_1^*(w)$ takes the form

$$H_1^*(w)=\frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{uH(u)}{u-w}\,du. \qquad (2.17)$$

Page 409.

As a result of (2.12) function uH(u) satisfies here the same inequality as H(u) in lemma II, and therefore occur and the inequalities indicated for $H_1^*(w)$ and $H_2^*(w)$

3. Formal solution of equation.

Let us return to equation (1.24), which let us write in the form

$$F(w)[1-K(w)]-G_1(w)=\Phi(w). \qquad (3.01)$$

In this expression function $\Phi(w)$ in any case is holomorphic in that region where all three functions F(w), G_1(w) and K(w) are holomorphic, i.e., within the band

$$0<\mathrm{Im}(w)\leqslant c', \qquad (3.02)$$

where c' - any positive number smaller than that by the constant c, which enters in (1.16).

Assuming in the formulas of lemma I a=0, b=c', $H(w)=\Phi(w)$ and using equality (1.23), we will ascertain that expansion (2.02) is reduced to the second term. This means that function $\Phi(w)$ in (3.01) is holomorphic not only within band (2.02), but also in the entire lower half-plane.

Thus, the solution of integral equation was reduced to the following task of the theory of functions.

They are given function K(w), the holomorphic in band (1.18), and function G_1(w), holomorphic in the upper half-plane. It is necessary to find this function F(w)so that itself it would be holomorphic in the upper half-plane, but entire expression (3.01) was

holomorphic not only in band (3.02), but also in the lower half-plane.

For the solution of this problem the vital importance has an expansion of function 1-K(w) into the factors. In any band, narrower than band (1.18), function K(w) is holomorphic not only inside, but also on the boundary; however, at infinity K(w) it vanishes. Therefore in this narrower band function 1-K(w) can have only finite number of roots.

Let us assume first that all these roots are complex. The case of real roots presents some special features/peculiarities and it will be examined separately. As a result of the parity of kernel k(x) function K(w) will be even, and the roots of function 1-K(w) will be arranged/located symmetrically relative to beginning.

Let us designate through v_0 distance from the real axis of the complex root nearest to it (as a result of the mentioned symmetry, such roots will be even a number). Let c* be a certain positively number less than v_0.

Page 410.

We can consider band

$$-c^* \leq \operatorname{Im}(w) \leq c^*, \qquad (3.03)$$

narrower, the initial band (1.18), and

possessing that property, that within it the function 1-K(w) does not have complex roots. In view of our assumption about the absence of real roots, this function will in no way have there roots. Therefore the function

$$\chi(w) = -\lg|1 - K(w)| \qquad (3.04)$$

will be within band (3.03) holomorphic. Since at infinity K(w) it approaches zero (conversely $|w|$), then it is possible to take this branch (main branch of logarithm)so that with Re(w) $\to +\infty$ there would be $\chi(w) \to 0$. But $\chi(w)$ - even function from w; therefore $\chi(w) \to 0$ also with Re(w) $\to -\infty$. In this case $\chi(w)$ will satisfy the inequality

$$|\chi(w)| < \frac{M}{|w|}. \qquad (3.05)$$

Thus, function $\chi(w)$ satisfies all conditions of lemma I, and we can decompose it on the sum

$$\chi(w) = \chi_1(w) + \chi_2(w) \qquad (3.06)$$

so that $\chi_1(w)$ would be holomorphic in band (3.03) and in the upper half-plane, and $\chi_2(w)$ - in the same band and in the lower half-plane. It is not difficult to see that as a result of parity $\chi(w)$

$$\chi_2(w) = \chi_1(-w). \qquad (3.07)$$

Passing from the logarithms to numbers and assuming/setting

$$e^{\chi_1(w)} = \psi_1(w), \quad e^{\chi_2(w)} = \psi_2(w), \qquad (3.08)$$

we we will obtain the following expansion of function 1-K(w) (it is more accurate, its reciprocal value) to the factors:

$$\frac{1}{1 - K(w)} = \psi_1(w)\psi_2(w). \qquad (3.09)$$

First factor $\psi_1(w)$ is holomorphic and does not have zero in band

(3.03) and in the upper half-plane, into second factor $\psi_2(w)$ it possesses the same properties in the same band and in the lower half-plane. As a result of (3.07)

$$\psi_2(w) = \psi_1(-w). \qquad (3.10)$$

At infinity functions χ_1 and χ_2 become zero, and functions ψ_1 and ψ_2 - in unity.

After obtaining expansion 1-K(w) into the factors, let us rewrite equation (3.01) in the form

$$\frac{F(w)}{\psi_1(w)\,\psi_2(w)} - G_1(w) = \Phi(w). \qquad (3.11)$$

Page 411.

After multiplying both parts (3.11) on $\psi_2(w)$, we will obtain

$$\frac{F(w)}{\psi_1(w)} - \psi_2(w)\,G_1(w) = \psi_2(w)\,\Phi(w). \qquad (3.12)$$

In the band

$$0 < \operatorname{Im}(w) < c^* \qquad (3.13)$$

[to upper half of band (3.03)] function $\psi_2(w)\,G_1(w)$ is holomorphic and satisfies the conditions of lemma I. It it is possible to decompose on the sum

$$\psi_2(w)\,G_1(w) = H_1(w) + H_2(w) \qquad (3.14)$$

so that the first term would be holomorphic in the upper half-plane, and the second term - in band (3.13) and in the lower half-plane, and so that each of them would become at infinity zero. Substituting (3.14) in (3.12) and transferring $H_2(w)$ into the right side, we will

obtain

$$\frac{F(w)}{\psi_1(w)} - H_1(w) = \psi_2(w)\Phi(w) + H_2(w). \qquad (3.15)$$

Here left side will be the function, holomorphic in the upper half-plane, whereas right side is holomorphic in band (3.13) and in the lower half-plane. Consequently, both parts of equality (3.15) are holomorphic in the entire plane, and it means, they are led to the constant. But this constant is equal to zero, since at infinity both parts (3.15) become zero. Therefore

$$F(w) = \psi_1(w) H_1(w), \qquad (3.16)$$

$$\Phi(w) = -\frac{H_2(w)}{\psi_2(w)}. \qquad (3.17)$$

Since in band (3.13) and in the lower half-plane function $\psi_2(w)$ not only is holomorphic, but also it does not have zero, then determined from (3.17) function $\Phi(w)$ will not have in this region of poles, but it will be in it actually/really holomorphic.

Thus, both functions (3.16) and (3.17) satisfy the conditions presented. We obtained, therefore, the solution of initial integral equation in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwx} F(w)\, dw. \qquad (3.18)$$

where F(w) has value (3.16).

Page 412.

4. Proof of the existence of integral for f(x).

In order to make our previous reasonings completely strict, we should demonstrate, on the basis of the properties of the assigned functions g(x) and k(x), that integral (3.18) for f(x) exists and that the presented to them function f(x) satisfies the integral equation proposed. Thereby will be proved the existence of solution.

Let us enumerate those conditions by which we must satisfy the assigned functions g(x) and k(x).

Function g(x). Function g(x):

by 1° it is absolutely integrated in the infinite gap/interval,

2° have in the infinite gap/interval the bounded variation.

From condition of 2° it follows that function g(x) approaches at infinity the specific limit, while from condition of 1° it follows that this limit is equal to zero.

From conditions of 1° and 2° escape/ensues the 1 applicability of Fourier's formula assumed in the paragraph. For the applicability of Fourier's formula it would be sufficient even require instead of

2° limitedness of variation in any final gap/interval. We will leave however in the force condition of 2°. In any case it will be carried out, if function g(x) has derivative g'(x), absolutely integrated in the infinite gap/interval.

Function k(x). Function k(x) we consider even. Let us subordinate to this its requirement: not only it itself, but also the function

$$k_1(x) = e^{c|x|}k(x) \qquad (c > 0) \tag{4.01}$$

positive by the constant c must satisfy conditions of 1° and 2°.

If functions g(x) and k(x) are complex, then we will consider that the conditions indicated satisfy their real and alleged parts separately.

Those superimposed to g(x) and k(x) of condition make it possible to, first of all, produce estimation for functions $G_1(w)$ and K(w). On the basis of this estimation, let us consider with the help of lemmas all II and III consecutively/serially functions, from which it is constructed by F(w). After obtaining estimation for F(w), we will be convinced of the existence of integral (3.18) for f(x), and then will demonstrate that this integral satisfies the equation proposed.

Let us designate through V total variation in function g(x) in gap/interval (0, ∞), if this function is real, or the sum of variations in its real and alleged part, if it is complex.

Page 413.

Then, on the basis of the second law of mean

$$\left|\int_0^\infty e^{iwx} g(x)\,dx\right| < \max\left|\int_0^\xi e^{iwx}\,dx\right| V, \qquad (4.02)$$

whence

$$|G_1(w)| < \frac{2}{|w|} V. \qquad (4.03)$$

This estimation will occur both on the real axis and in the entire upper half-plane.

After representing value K(w) in the form

$$K(w) = \int_0^\infty (e^{iwx} + e^{-iwx})\, e^{-\varepsilon x} k_1(x)\,dx, \qquad (4.04)$$

we will obtain analogous estimation, also, for it. Let V_1 be the sum of total variation in the real and imaginary part of function $k_1(x)$. Then

$$|K(w)| < \max\left|\int_0^\xi (e^{iwx} + e^{-iwx})\, e^{-\varepsilon x}\,dx\right| V_1, \qquad (4.05)$$

whence

$$|K(w)| < V_1 \;\; U(w), \qquad (4.06)$$

where

$$U(w)=\frac{1}{|w+ic|}+\frac{1}{|w-ic|}+\frac{2c}{|w+ic||w-ic|}. \qquad (4.07)$$

With the sufficiently large $|w|$

$$|K(w)|<\frac{2V_1'}{|w|}, \qquad (4.08)$$

where V'_1 - any number, greater V_1.

Inequality (4.06) will be correct for the entire band

$$-c\leqslant \mathrm{Im}(w)\leqslant c, \qquad (4.09)$$

switching on its boundaries, whereas inequality (4.08) - only in that part of band (4.09), which is sufficiently distant from the imaginary axis.

Limited by formula (3.04) function $\chi(w)$ with the sufficiently large $|w|$ will satisfy the analogous inequality

$$|\chi(w)|<\frac{M}{|w|}, \qquad (4.10)$$

in which it is possible to take $M=2V'_1$ by the previous value of V'_1. This inequality will occur both within band (4.09)and on its boundary.

Page 414.

Considering on the basis of lemma II both components/terms/addends in (3.06), we obtain

$$|\chi_1(w)|<\frac{M_1}{|w|}\lg|w|, \quad |\chi_2(w)|<\frac{M_1}{|w|}\lg|w|, \qquad (4.11)$$

where M_1 - new constant. Hence for function $\psi_1(w)$ we obtain the inequality

$$|\psi_1(w)-1|<\frac{M_1'}{|w|}\lg|w|, \qquad (4.12)$$

valid for the sufficiently large $|w|$ on the real axis and in the upper half-plane. The same inequality satisfies function $\psi_2(w)$.

Passing to formula (3.14), we will use lemma I not to function $\psi_2(w)G_1(w)$, and to function $[\psi_2(w)-1]G_1(x)$. Resolution this latter will, obviously, take the form

$$[\psi_2(w)-1]G_1(w)=[H_1(w)-G_1(w)]+H_2(w), \qquad (4.13)$$

since the subtrahend $G_1(w)$ wholly relates to the first member of right side.

By force (4.03) and (4.12) the function on the left side (4.13) will be satisfy to the inequality

$$|(\psi_2-1)G_1|<\frac{2M_1'V}{|w|^2}\lg|w|, \qquad (4.14)$$

but it means, and the inequality

$$|(\psi_2-1)G_1|<\frac{M_\sigma}{|w|^{1+\sigma}}, \qquad (4.15)$$

where σ - any positive number smaller than one.

On the basis of lemma III

$$H_1(w)-G_1(w)=\frac{iC}{w}+\frac{1}{w}H_1^*(w), \qquad (4.16)$$

$$H_2(w)=-\frac{iC}{w}+\frac{1}{w}H_2^*(w); \qquad (4.17)$$

the constant C can be expressed by the absolutely convergent integral

$$C = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\psi_2(w) - 1]\,G_1(w)\,dw, \qquad (4.18)$$

but function $H_1^*(w)$ and $\overline{H_2^*}(w)$ according to lemma III they vanish as $|w|^{-\sigma'}$, where σ' - any positive number smaller than one.

Page 415.

After obtaining estimation $H_1(w)$ and $H_2(w)$, we can switch over to estimation F(w) and $\Phi(w)$, which are expressed as $H_1(w)$ and $H_2(w)$ according to formulas (3.16) and (3.17). The first of these formulas let us write in the form

$$F(w) = \dot{H}_1(w) + [\psi_1(w) - 1]\,\dot{H}_1(w). \qquad (4.19)$$

Hence, using (4.16), we obtain

$$F(w) = G_1(w) + \frac{iC}{w} + F^*(w), \qquad (4.20)$$

F* (w) with the the large |w| satisfies the inequality

$$|F^*(w)| < \frac{M_\sigma^*}{|w|^{1+\sigma}} \quad (0 < \sigma < 1). \qquad (4.21)$$

Estimation for $\Phi(w)$ can be obtained both from (4.17) and (3.17)and from (4.20) and (3.01). That, etc. gives

$$\Phi(w) = \frac{iC}{w} + \Phi^*(w), \qquad (4.22)$$

moreover with the sufficiently large |w|

$$|\Phi^*(w)| < \frac{M_\sigma^*}{|w|^{1+\sigma}} \quad (0 < \sigma < 1). \qquad (4.23)$$

with the same values σ and M_σ^*, as in (4.21).

Formulas (4.20) and (4.22) have the inconvenience, that the entering in them functions F^* and Φ^* go to infinity with w=0. This inconvenience is easily reduced. Let us introduce the arbitrary number q with the positive imaginary part (it is possible to take, for example, $q=iv_0$, where v_0 is defined as in paragraph 3). Instead of (4.20) and (4.22) we can write

$$F(w) = G_1(w) + \frac{iC}{w+q} + F_q^*(w), \qquad (4.24)$$

$$\Phi(w) = \frac{iC}{w-q} + \Phi_q^*(w). \qquad (4.25)$$

Here F_q^* and Φ_q^* satisfy the same inequalities as F^* and Φ^*, and, furthermore, they are the holomorphic functions: the first - in the upper half-plane, and the second - in band (3.03) and in the lower half-plane.

After obtaining $F_q^*(w)$, it is possible to construct the absolutely convergent integral

$$f_q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwx} F_q^*(w)\, dw. \qquad (4.26)$$

Page 416.

This integral will be the limited and continuous function of x, which on Riemann-Lebesgue's lemma will vanish with $x \to \infty$. Furthermore,

$$f_q(x) = 0 \ \overset{(1)}{\text{при}} \ x \leqslant 0. \qquad (4.27)$$

Key: (1). with.

Function g(x), but to assumption, satisfies formulated above conditions 1° and 2°, so that to it is applicable Fourier's formula. Therefore

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{+N}e^{-ixw}G_1(w)\,dw=\begin{cases}g(x) & \overset{(1)}{\text{при}}\ x>0,\\ 0 & \text{при}\ x<0.\end{cases}\qquad(4.28)$$

Key: (1). with.

Furthermore,

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{+N}e^{-ixw}\frac{idw}{w+q}=\begin{cases}e^{iqx} & \overset{(1)}{\text{при}}\ x>0,\\ 0 & \text{при}\ x<0.\end{cases}\qquad(4.29)$$

Key: (1). with.

From the comparison of formulas (4.26), (4.28) and (4.29) follows that there is an integral

$$f(x)=\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{+N}e^{-ixw}F(w)\,dw,\qquad(4.30)$$

that being determining function f(x), moreover this integral is equal to

$$f(x)=\begin{cases}g(x)+Ce^{iqx}+f_q(x) & \overset{(1)}{\text{при}}\ x>0,\\ 0 & \text{при}\ x<0.\end{cases}\qquad(4.31)$$

Key: (1). with.

The determined by it function f(x) is the sum of continuous function and function with the bounded variation; function f(x) is everywhere limited and approaches at infinity zero (let us recall that Im(q)>0).

The conditions, necessary so that to the function f(x) would be applicable Fourier's formula, will be established/installed in paragraph 8.

5. Proof of the existence of solution

After demonstrating the existence of integral for f(x), we arrive at the second aspect of our task let us demonstrate that the determined by it function f(x) satisfies the integral equation proposed.

Page 417.

Let us demonstrate, first of all, formula (1.13). For this us will be required one property of the integral

$$f_N(x)=\frac{1}{2\pi}\int_{-N}^{+N} e^{-ixw}F(w)\,dw, \qquad (5.01)$$

above which it is completed in (4.30) the passage to the limit, namely its uniform limitedness both relative to N and relative to x.

Let us consider first the integral

$$g_N(x)=\frac{1}{2\pi}\int_{-N}^{+N} e^{-ixw}G_1(w)\,dw \qquad (5.02)$$

and let us demonstrate uniform limitedness for it. As a result of the absolute integrability of function g(x) we can in the integral

$$\int_0^\infty e^{iw(y-x)}g(y)\,dy=e^{-ixw}G_1(w) \qquad (5.03)$$

integrate under the integral sign, after which we will obtain

$$g_N(x)=\frac{1}{\pi}\int_0^\infty \frac{\sin N(x-y)}{x-y}g(y)\,dy. \qquad (5.04)$$

After using the second law of mean, it is easy to consider integral in the right side. Specifically,

$$\left|\frac{1}{\pi}\int_0^\eta \frac{\sin N(x-y)}{x-y}dy\right|\leqslant\frac{2}{\pi}\int_0^\pi\frac{\sin t}{t}dt, \qquad (5.05)$$

moreover

$$\frac{2}{\pi}\int_0^\pi\frac{\sin t}{t}dt=1{,}17898\ldots<1{,}18. \qquad (5.06)$$

Equality (5.05) proves to be valid, whatever N, x and η. Designating through V total variation in function g(x) in gap/interval (0, ∞) and noting that at infinity g(x)=0.

$$|g_N(x)|<1{,}18V. \qquad (5.07)$$

that also proves the uniform limitedness of integral (5.02).

Assuming/setting in previous formulas $g(x)=e^{iqx}$, moreover Im(q)>0,

we consist that also the integral, which stands under the sign of limit in formula (4.29), uniformly bounded relative to N and x.

Page 418.

Finally, the integral

$$f_{qN}(x)=\frac{1}{2\pi}\int_{-N}^{+N} e^{-iwx}F_q^*(w)\,dw \tag{5.08}$$

can be evaluated on the modulus/module of integrand, which gives

$$|f_{qN}(x)|<\frac{1}{2\pi}\int_{-\infty}^{\pm\infty}|F_q^*(w)|\,dw, \tag{5.09}$$

independent of N and of x.

Comparing the obtained estimations for three addend formula (4.31), we consist that also integral (5.01) uniformly bounded, QED.

We pass to the proof of formula (1.13). As a result of the absolute integrability of function k(x-y) the integral

$$I(w)=\int_{-\infty}^{+\infty} k(x-y)\,e^{-iwy}\,dy \tag{5.10}$$

descends evenly relative to w (which is assumed to be here real). Therefore in the integral

$$\frac{1}{2\pi}\int_{-N}^{+N} I(w)\,F(w)\,dw=\frac{1}{2\pi}\int_{-N}^{+N} e^{-iwx}K(w)\,F(w)\,dw \tag{5.11}$$

it is possible to integrate under the integral sign which gives to us

$$\frac{1}{2\pi}\int_{-N}^{+N} e^{-iwx}K(w)F(w)\,dw =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} k(x-y)\,dy \int_{-N}^{+N} e^{-iwy}F(w)\,dw. \qquad (5.12)$$

Here it is possible to pass to limit of $N \to \infty$. The limit of left side exists as a result of the absolute convergence of the integral standing there. Right side can be according to (5.01) registered in the form

$$\int_{-\infty}^{+\infty} k(x-y)f_N(y)\,dy = \int_{-\infty}^{+\infty} k(x-y)f(y)\,dy + R_N(x), \qquad (5.13)$$

where

$$R_N(x) = \int_{-\infty}^{+\infty} k(x-y)\,[f_N(y) - f(y)]\,dy. \qquad (5.14)$$

It is easy to see that

$$\lim_{N\to\infty} R_N(x) = 0. \qquad (5.15)$$

Page 419.

In fact, in integral (5.14) the way of integration can be divided into three sections: from $-\infty$ to x-A, from x-A to x+A and from x+A to $+\infty$. Of that proved, function $f_N(y)$ and that means, also difference $f_N(y) - f(y)$ are uniformly bounded. Therefore as a result of the absolute integrability of function k(x-y) it is possible to select A (not depending on N) by so/such large that the sum of integrals on the extreme sections would be how the angular-bottom of small. After

this, using limitedness $f_N(y)$, it is possible to select N by so/such large that the integral on the middle section would be as small as desired (according to the Lebeg theorem). Then also the entire integral (5.14) would be as small as desired. Thereby equality (5.15) is proved.

Thus, we demonstrated the equality

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwx} K(w) F(w)\, dw = \int_{-\infty}^{+\infty} k(x-y) f(y)\, dy, \qquad (5.16)$$

i.e. formula (1.13). Let us note that in our reasonings we did not use assumption about the fact that to f(x) was applicable Fourier's formula (this assumption by us thus far and is not proved).

Since f(y) with negative y with negative y is equal to zero, then instead of (5.16) it is possible to write

$$\int_{0}^{\infty} k(x-y) f(y)\, dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwx} K(w) F(w)\, dw. \qquad (5.17)$$

But F(w) satisfies the functional equation according to which

$$K(w) F(w) = [F(w) - G_1(w)] - \Phi(w). \qquad (5.18)$$

Understanding integral in right side (5.17) as the limit of integral from -N to +N, we can represent it in the form of a difference in two integrals whose existence escape/ensues from estimations (4.24) and (4.25) and which with x>0 are equal to

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{+N} e^{-iwx}\,[F(w)-G_1(w)]\,dw = f(x)-g(x). \quad (5.19)$$

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{+N} e^{-iwx}\Phi(w)\,dw = 0. \quad (5.20)$$

During the comparison of latter/last four formulas we obtain

$$\int_{0}^{\infty} k(x-y)\,f(y)\,dy = f(x)-g(x). \quad (5.21)$$

i.e. the integral equation proposed.

Page 420.

We demonstrated that with those done relative to g(x) and k(x) the assumptions there is a solution of our integral equation, represented in the form of integral (4.30). Some properties of function (4.30) have noted we above: this - sum from the function with the bounded variation and the continuous function. Solution (4.30) remains everywhere final and at infinity vanishes.

Subsequently (paragraph 10) it will be shown that solution (4.30) will be unique solution with these properties.

On the other hand, if we superimpose on g(x) the additional requirement of the absolute integrability x g(x), then it is possible to claim that to the obtained solution is applicable Fourier's

formula (this will be shown in paragraph 11). Then from the very method of the conclusion/output of our solution it is clear that it will also be the unique solution to which this formula is applicable.

As far as the limitations, superimposed by us on g(x) and k(x), are concerned, it is not difficult to see that they can be moderated. Thus, for instance, retaining them completely for g(x), we can allow inversion k(x) with x=0 in infinity order $|x|^{\rho-1}$, $\rho>0$ so that the difference

$$k(x)-A|x|^{\rho-1}e^{-c|x|}=k_{1}(x) \qquad (5.22)$$

would remain final and would satisfy all previous conditions. This case frequently is encountered in the physical tasks. Then for K(w) will occur the estimation

$$|K(w)|\leq\frac{M_{\rho}}{|w|^{\rho}} \text{ при}^{(1)} |w|\to\infty \qquad (5.23)$$

Key: (1). with.

and all conclusions of this and previous of paragraphs will remain in the force. Similar to this it is possible to also allow inversion k(x) into logarithmic infinity with x=0. Are possible other softenings of the conditions, assigned on the assigned functions k(x) and g(x), but on them we stop will not be.

6. Case of real roots.

In the previous analysis we left aside the case of the real roots of function 1-K(w). Let us produce one conversion of equation (3.01), applied also in the general case, but especially convenient for studying the case of real roots. About the matter of roots for the time being we will not make assumptions.

Page 421.

Let the function 1-K(w) in the band

$$-c < \operatorname{Im}(w) < +c, \qquad (6.01)$$

where it is holomorphic, have 2n roots, from which n they are equal to:

$$w = w_1, \quad w = w_2, \ldots, \quad w = w_n, \qquad (6.02)$$

and the rest n differ from them it is familiar. Among roots (6.02) there can be identical. Let us agree to consider that those of roots (6.02) which are complex, have the positive imaginary part and are arranged/located in the ascending order of their imaginary part.

Let us introduce positive number b>c, where c - half of width of band (6.01), and let us assume

$$\frac{1}{1-K(w)} = \frac{(w^2+b^2)^n}{(w^2-w_1^2)\ldots(w^2-w_n^2)}\psi(w). \qquad (6.03)$$

Function $\psi(w)$ determined by this equality is holomorphic and does not have zero within band (6.01), but at infinity it is converted into unity. Its logarithm $\chi = \lg\psi$ will be within the band

holomorphic function which at infinity it vanishes conversely |w|. This function satisfies the conditions of lemma I, and we can use to it the resolution of form (3.06), which leads to expansion $\psi(w)$ into the factors

$$\psi(w) = \psi_1(w)\,\psi_2(w) \qquad (6.01)$$

with the properties, analogous to the properties of factors in formula (3.09).

Substituting (6.04) in (6.03), we will obtain

$$\frac{1}{1-K(w)} = \frac{(w^2+b^2)^n\,\psi_1(w)\,\psi_2(w)}{(w^2-w_1^2)\ldots(w^2-w_n^2)}. \qquad (6.05)$$

After assuming

$$\psi_{n1}(w) = (w+ib)^n\,\psi_1(w), \qquad (6.06)$$

$$\psi_{n2}(w) = (w-ib)^n\,\psi_2(w), \qquad (6.07)$$

we we can also write

$$\frac{1}{1-K(w)} = \frac{\psi_{n1}(w)\,\psi_{n2}(w)}{(w^2-w_1^2)\ldots(w^2-w_n^2)}. \qquad (6.08)$$

Let us note that the form of the function $\psi_1(w)$ and $\psi_2(w)$ depends on the selection of the arbitrary number b, whereas expressions $\psi_{n1}(w)$ and $\psi_{n2}(w)$ do not depend on b.

Above we assumed that the complex roots from series/row (6.02) were arranged/located in the ascending order of their alleged part. If we instead of band (6.01) examined the narrower band

$$-c' < \mathrm{Im}(w) < +c' \qquad (0 < c' < c), \qquad (6.09)$$

then it would not hit some roots, for example roots with marks n'+1, n'+2 ..., n(n'<n). Formula (6.05) would take the form

$$\frac{1}{1-K(w)}=\frac{(w^2+b^2)^{n'}\psi_1^*(w)\,\psi_2^*(w)}{(w^2-w_1^2)\dots(w^2-w_{n'}^2)}. \qquad (6.10)$$

Page 422.

Is easy to establish/install connection/communication between $\psi_1^*(w)$ and $\psi_1(w)$, also between $\psi_2^*(w)$ and $\psi_2(w)$. Specifically,

$$\psi_1^*(w)=\frac{(w+ib)^{n-n'}\,\psi_1(w)}{(w+w_{n'+1})\dots(w+w_n)},\qquad (6.11)$$

$$\psi_2^*(w)=\frac{(w-ib)^{n-n'}\,\psi_2(w)}{(w-w_{n'+1})\dots(w-w_n)}.\qquad (6.12)$$

(Let us recall that the alleged part of complex roots w_m we consider positive).

By a similar contraction of band we always can achieve that it would hit only real roots. Therefore there would not be by the limitation of generality and the assumption that all roots (6.02) were real.

We pass to the solution of the functional equation

$$F(w)\,[1-K(w)]-\bar{G}_1(w)=\Phi(w).\qquad (6.13)$$

Let us substitute in it 1-K(w) from (6.05) and will multiply both

parts of the obtained equation on $\psi_2(w)$. Then

$$\frac{F(w)}{\psi_1(w)}\frac{(w^2-w_1^2)\ldots(w^2-w_n^2)}{(w^2+b^2)^n}-\psi_2(w)G_1(w)=\psi_2(w)\Phi(w). \quad (6.14)$$

Function $\psi_2(w)G_1(w)$as in paragraph 3 [formula (3.14)], it is possible to decompose according to the formula

$$\psi_2(w)G_1(w)=H_1(w)+H_2(w). \quad (6.15)$$

If we introduce the designations:

$$\frac{F(w)}{\psi_1(w)}=F^*(w), \quad (6.16)$$

$$H_1(w)=G_1^*(w), \quad (6.17)$$

$$\psi_2(w)\Phi(w)+H_2(w)=\Phi^*(w), \quad (6.18)$$

the equation (6.14) signs the form

$$F^*(w)\frac{(w^2-w_1^2)\ldots(w^2-w_n^2)}{(w^2+b^2)^n}-G_1^*(w)=\Phi^*(w). \quad (6.19)$$

The introduced recently functions F^* , $G_1{}^*$, and Φ^* possess the properties, completely analogous to the properties of function F, G_1 and Φ.

Page 423.

Thus, the general case of functional equation (6.13) is given to the special case, when the converted kernel $K^*(w)$ is fractional rational integral function from w form

$$K^*(w) = 1 - \frac{(w^2 - w_1^2)\dots(w^2 - w_n^2)}{(w^2 + b^2)^n}. \qquad (6.20)$$

However, for the kernel (6.20) the task can be solved by purely algebraic path.

Let us produce the substitution

$$\frac{w - ib}{w + ib} = z, \qquad (6.21)$$

which transforms the upper half-plane w into the internal part of the circle of a single radius, and the lower half-plane w - into the external part of the same circle. To the infinite point of plane w will correspond point z=1. Any function from w, holomorphic in the upper half-plane, will be decomposed/expanded in the series/row according to the positive degrees of z, which converges with $|z|<1$. But the function, holomorphic in the lower half-plane, will decompose in the series/row according to negative degrees of z, which converges with $|z|>1$.

It is obvious that, after expressing function (6.20) through z, we will obtain polynomial from z and from 1/z (cuttings off of Laurent series). We can assume

$$1 - K^*(w) = L(z), \qquad (6.22)$$

where

$$L(z) = \sum_{m=-n}^{+n} c_m z^m \qquad (c_{-m} = c_m). \qquad (6.23)$$

With this L(1)=1. Let us assume, further,

$$F^{*}(w) = F_3(z), \qquad F_3(1) = 0, \tag{6.24}$$

$$G_1^{*}(w) = G_3(z), \qquad G_3(1) = 0, \tag{6.25}$$

$$\Phi^{*}(w) = \Phi_3\left(\frac{1}{z}\right), \quad \Phi_3(1) = 0. \tag{6.26}$$

function F_3, G_3, Φ_3 will be ascending series from their arguments. Let us substitute these expressions in equation (6.19). Then

$$F_3(z)\sum_{m=-n}^{+n} c_m z^m - G_3(z) = \Phi_3\left(\frac{1}{z}\right). \tag{6.27}$$

But left side contains only a finite number of negative degrees z (lowest degree - minus n). Therefore right side will be the polynomial of degree n of 1/z.

Page 424.

It is possible to assume

$$\Phi_3\left(\frac{1}{z}\right) = z^{-n} Q_n(z), \tag{6.28}$$

where $Q_n(z)$ – polynomial of degree n from z.

Deciding (6.27) relative to F_3, we obtain

$$F_3(z) = \frac{z^n G_3(z) + Q_n(z)}{z^n L(z)}. \tag{6.29}$$

Singular points $F_3(z)$ within the unit circle could be the lying/horizontal there roots L(z). But function $F_3(z)$ must not have there singular points. When task has a solution, this requirement makes it possible to determine unknown polynomial $Q_n(z)$.

Here it is necessary to distinguish two cases: the first case when all roots w_m are complex, and the second case when some of them are real.

Case 1. Let us assume that all values (6.02) have positive alleged part. Then the corresponding to them values z

$$z = z_1, \quad z = z_2, \ldots, \quad z = z_n \tag{6.30}$$

will be everything on to modulus/module less than unity. Values (6.30) will be the only roots of L(z), which lie within the unit circle, but on the circle itself this function will not have roots. [Rest n of roots L(z) they will be equal to reciprocal values (6.30) and they lie/rest out of the circle].

So that fraction (6.29) would remain within the circle of final, the numerator of fraction must become zero for values (6.30). This it gives for polynomial $Q_n(z)$ n the conditions which in the case of simple roots take the form

$$Q_n(z_m) = -z_m^n G_3(z_m) \quad (m = 1, 2, \ldots, n). \tag{6.31}$$

With the multiple roots into the conditions will enter the derivatives of $Q_n(z)$ and $G_3(z)$. If root $z = z_m$ of multiplicity s, then the conditions corresponding to it will be

$$Q_n^{(r)}(z_m) = -\left\{\frac{d^r}{dz^r}\left[z^n G_3(z)\right]\right\}_{z=z_m} (r = 0, 1, \ldots, s-1). \quad (6.32)$$

Both in that and in other case the total number of conditions will be equal to n. These n of conditions together with the condition

$$Q_n(1) = 0 \quad (6.33)$$

determine polynomial $Q_n(z)$ in a single-valued manner.

Page 425.

We arrived at the result, already established/installed in the previous paragraphs by other means: in the case of complex roots 1-K(w) the solution exists for any function g(x), which satisfies some general conditions and for by any appropriate by her $G_3(x)$.

Case 2. Let us assume that the function 1-K(w) has 21 the real roots

$$w = w_1, w = w_2, \ldots, w = w_l, \quad (6.34)$$

and also

$$w = -w_1, w = -w_2, \ldots, w = -w_l. \quad (6.35)$$

However, the remaining roots

$$w = w_{l+1},\ w = w_{l+2}, \ldots, w = w_n \qquad (6.36)$$

and the values, which differ from them it is familiar, let be complex [1].

FOOTNOTE [1]. According to the observation done above, we could consider that l=n, i.e., then complex roots (6.36) are absent. ENDFOOTNOTE.

In similar of the case of first l of values (6.30) they will be on the modulus/module equal to unity. A number of points for which it is necessary to require inversion into zero numerators of fraction (6.29), will be (without considering point z=1) equal to n+l, since to points (6.30) it is necessary to join the points

$$z = \frac{1}{z_1},\quad z = \frac{1}{z_2}, \ldots, z = \frac{1}{z_l}, \qquad (6.37)$$

corresponding to roots (6.36). A number of conditions, assigned on the coefficients of the polynomial $\bar{Q}_n(\bar{z})$. will exceed a number of coefficients on l. So that these conditions would be combined, it is necessary that the assigned function $G_3(z)=G_1^*(w)$ would satisfy several relationships/ratios which connect the values of this function (but in the case of multiple roots - value of its derivatives) at different points z= z_m and respectively w=w_m. A number of such relationships/ratios is equal to l - to number of pairs of real roots.

For initially assigned function g(x) the relationships/ratios indicated are led to the conditions of the orthogonality

$$\int_0^\infty g(x)\,\gamma_m(x)\,dx = 0 \quad (m = 1, 2, \ldots, l), \tag{6.38}$$

where the form of the function $\gamma_m(x)$ depends only on kernel k(x).

These functions, which represent the solutions of homogeneous integral equation, will be by us studied in paragraphs 8 and 9.

Page 426.

In our formulas we can return from by the variable/alternating z to by the variable/alternating w. From formulas (6.26) and (6.28) we obtain

$$\Phi^*(w) = \frac{P_{n-1}(w)}{(w - ib)^n}, \tag{6.39}$$

where $P_{n-1}(w)$ — the polynomial of degree n-1 from w.

Hence

$$F^*(w) = \frac{(w^2 + b^2)^n G_1^*(w) + (w + ib)^n P_{n-1}(w)}{(w^2 - w_1^2) \ldots (w^2 - w_n^2)}. \tag{6.40}$$

Passing now according to formulas (6.16) and (6.17) from F* (w) and G_1* (w) to the initial functions F(w) and H_1(w), we will have

$$F(w) = \psi_1(w) \frac{(w^2 + b^2)^n H_1(w) + (w + ib)^n P_{n-1}(w)}{(w^2 - w_1^2) \ldots (w^2 - w_n^2)} \tag{6.41}$$

or, if we will use designation (6.06),

$$F(w)=\psi_{n1}(w)\frac{(w-ib)^n H_1(w)+P_{n-1}(w)}{(w^2-w_1^2)\dots(w^2-w_n^2)}. \qquad (6.42)$$

Let us recall that function $\psi_{n1}(w)$ [in contrast to ψ_1 (w)] it does not depend on the selection of number b. As far as the numerator of fraction in (6.42)is concerned, it also does not depend on b. In fact, according to formula (6.15) we have according to lemma I with Im(w)>0

$$H_1(w)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\psi_2(u)\,G_1(u)}{u-w}\,du. \qquad (6.43)$$

In order to express explicitly dependence of H_1 on b, let us write the previous formula in the form

$$H_1(w)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\psi_{n2}(u)\,G_1(u)}{(u-ib)^n(u-w)}\,du. \qquad (6.44)$$

Let us designate through $H'_1(w)$ the function which is obtained from $H_1(w)$ by replacement of b to certain another number b', then

$$(w-ib)^n H_1(w)-(w-ib')^n H'_1(w)=$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\left(\frac{w-ib}{u-b}\right)^n-\left(\frac{w-ib'}{u-ib'}\right)^n\right]\frac{\psi_{n2}(u)\,G_1(u)}{u-w}\,du. \qquad (6.45)$$

But this is the polynomial of degree n-1 from w which can be connected in P_{n-1} (w).

Page 427.

Polynomial P_{n-1} (w) is determined from the condition, according to which the numerator of fraction (6.42) must become zero at those points of the upper half-plane in which the denominator is converted into zero.

In the case of complex roots these conditions take the form

$$P_{n-1}(w_m) = -(w_m - ib)^n H_1(w_m), \qquad (6.46)$$

if roots are simple, and somewhat the more complicated form, which contains derivatives, if roots are multiple. A number of such conditions is equal to n.

In the case of l of the pairs of real roots to these n to conditions they are adjoined l of the new conditions of analogous form, and from them they escape/ensue for the values of function H_1(w) and its derived at points w= $w_m l$ relationships/ratios, which guarantee the consistency of the mentioned conditions for $P_{n-1}(w)$. If these relationships/ratios are fulfilled, function F(w) will not have poles not only in the upper half-plane, but also on the real axis.

7. Further conditions for the assigned function in the case of real roots.

In paragraphs 4 and 5 we demonstrated the existence of solution in the form of integral (3.18) for the case of complex roots. Now we

should explain, what further limitations (without considering the conditions of orthogonality) they must be superimposed to g(x)so that the solution of the same form would exist also in the case of real roots.

These limitations must be such that from them would escape/ensue the absolute integrability F(w) near each root of w= w_m.

For function F(w) we can use expression (6.42), in which to more conveniently replace $H_1(w)$ with his value

$$H_1(w) = \psi_2(w) G_2(w) - H_2(w), \qquad (7.01)$$

for those undertaken from (3.14). Therefore for F(w) is obtained an expression which, of course, is equivalent to the formula

$$F(w) = \frac{G_1(w) + \Phi(w)}{1 - K(w)}, \qquad (7.02)$$

of that escape/ensuing directly from the fundamental functional equation for F(w).

In expression (7.02) functions K(w) and Φ (w), according to their definition, are holomorphic in the band, which switches on the real axis [and $\Phi_1(w)$, furthermore, and in the lower half-plane].

Page 428.

Concerning $G_1(w)$, this function is holomorphic only above the real

axis, and on axis itself it [under the done, until now, assumptions relative to g(x)] is only continuous and limited. Therefore behavior F(w) on the real axis is determined by behavior $G_1(w)$. Let w= w_m be the root of the denominator of multiplicity s. Near w= w_m

$$F(w) = A(w) D_s(w) + B(w), \qquad (7.03)$$

where

$$D_s(w) = (w - w_m)^{-s} \Big[G_1(w) - G_1(w_m) - \\ - (w - w_m) G_1'(w_m) - \cdots - \frac{(w - w_m)^{s-1}}{(s-1)!} G_1^{(s-1)}(w_m) \Big], \quad (7.04)$$

but functions A(w) and B(w) are holomorphic near w= w_m.

If root w= w_m were complex (with a positive imaginary part), then near $\bar{w} = w_m$ function $G_1(w)$, and also, therefore, $D_s(w)$, , would be analytical. Consequently, holomorphy F(w) near the complex ones, at least and multiple, the roots of denominator it will occur without any additional limitations for g(x).

But if root $w = w_m$ is real, then the requirement of the absolute integrability F(w) near w= w_m assigns on $G_1(w)$ and on g(x) some new limitations. This requirement will be met, if value $D_s(w)$ proves to be absolutely integrated.

Let us assume for certainty $w \geqslant w_m$ (we we count here w real) let us demonstrate the inequality

$$\int_{w_m}^{w} |D_s(w)|\, dw \leqslant \frac{1}{(s-1)!} \int_{w_m}^{w} \left| \frac{G_1^{(s-1)}(u) - G_1^{(s-1)}(w_m)}{u - w_m} \right| du. \quad (7.05)$$

(With $w \leqslant w_m$ it was necessary to take here w as the lower and w_m as the upper limit).

Formula (7.05), obviously, is valid for s=1, since in this case it becomes identical. In order to demonstrate it for $s \geq 2$, we will use representation for $D_s(w)$ in the form of the integral

$$D_s(w) = \int_0^1 \left(\frac{G_1^{(s-1)}[wt + w_m(1-t)] - G_1^{(s-1)}(w_m)}{w - w_m} \right) \frac{(1-t)^{(s-2)}}{(s-2)!} dt. \tag{7.06}$$

Let us assume

$$I(t) = \int_{w_m}^{w} \left| \frac{G_1^{(s-1)}[wt + w_m(1-t)] - G_1^{(s-1)}(w_m)}{w - w_m} \right| dw. \tag{7.07}$$

Page 429.

Value I(t) can be represented in the form

$$I(t) = \int_{w_m}^{w'} \left| \frac{G_1^{(s-1)}(u) - G_1^{(s-1)}(w_m)}{u - w_m} \right| du, \tag{7.08}$$

where

$$w' = wt + w_m(1-t). \tag{7.09}$$

Let us note that t enters into expression (7.08) only by means the

upper limit.

If we assume the absolute integrability of the function

$$\Delta(u, v) = \frac{G_1^{(s-1)}(u) - G_1^{(s-1)}(v)}{u - v}, \qquad (7.10)$$

then integral I(t) will be the limited and continuous function of t, moreover

$$I(t) \leqslant I(1) = \int_{w_m}^{w} |\Delta(u, w_m)|\, du, \qquad (7.11)$$

since, with $t \leq 1$ there will be $w' \leq w$.

Therefore we after multiplying I(t) on $\frac{(1-t)^{s-2}}{(s-2)!}dt$, to integrate under the integral sign; as a result we will obtain

$$\int_{w_m}^{w} dw \left\{ \int_0^1 \left| \frac{G_1^{(s-1)}[wt + w_m(1-t)] - G_1^{(s-1)}(w_m)}{w - w_m} \right| \frac{(1-t)^{s-2}}{(s-2)!} dt \right\} = \int_0^1 I(t) \frac{(1-t)^{s-2}}{(s-2)!} dt. \qquad (7.12)$$

Internal integral on the left side (7.12) gives as a result of (7.06) the upper limit for $|D_s(w)|$. Entire/all left side (7.12) gives the upper imit for left side (7.05). Therefore

$$\int_{w_m}^{w} |D_s(w)|\, dw \leqslant \int_0^1 I(t) \frac{(1-t)^{s-2}}{(s-2)!} dt. \qquad (7.13)$$

Substituting here by its I(t) upper limit from (7.11) and implementing integration for t, we will obtain

$$\int_{w_m}^{w} |D_s(w)|\, dw \leqslant \frac{1}{(s-1)!} \int_{w_m}^{w} |\Delta(u, w_m)|\, du, \qquad (7.14)$$

i.e. formula (7.05).

Page 430.

Thus, so that the function F(w) would be absolutely integrated near point $w = w_m$, it suffices to require the absolute integrability of function $\Delta(w, w_m)$ near this point.

Let us place in expression (7.10) for Δ(u, v) number s equal to the greatest multiplicity of real roots. If for this value of s value Δ(u, v) will be the absolutely integrated function from w at any value of u, then, obviously, and F(w) will be that absolutely integrated in any final gap/interval.

Entering the expression for Δ(u, w) derivative is equal to

$$G_1^{(s-1)}(w) = i^{s-1} \int_0^{\infty} e^{ixw} x^{s-1} g(x)\, dx. \qquad (7.15)$$

The existence of this derivative and the convergence of integral (7.15) are necessary already (as we will see in paragraph 8) for the formulation of the conditions of orthogonality (6.38), assigned on g(x). We will require, furthermore so that integral (7.15) would be that absolutely converging, and let us assume

$$h(x) = \int_x^\infty x^{s-1} |g(x)|\, dx. \quad (7.16)$$

Let us demonstrate that the function Δ(u, w) satisfies the inequality

$$|\Delta(u, w)| \leqslant \int_0^\xi h(x)\, dx, \quad (7.17)$$

where

$$\xi = \frac{2}{|u-w|}. \quad (7.18)$$

We have

$$G_1^{(s-1)}(w) - G_1^{(s-1)}(u) = i^{s-1} \int_0^\infty (e^{ixw} - e^{ixu})\, x^{s-1} g(x)\, dx. \quad (7.19)$$

To a difference in the exponential functions under the integral we apply the inequalities

$$|e^{ixw} - e^{ixu}| = \left|2 \sin \frac{x(u-w)}{2}\right| < x|u-w| \text{ при}^{(1)} \ x < \xi, \quad (7.20)$$

$$|e^{ixw} - e^{ixu}| \leqslant 2 \quad \text{при}^{(1)} \ x > \xi. \quad (7.21)$$

Key: (1). with.

Then

$$|G_1^{(s-1)}(u) - G^{(s-1)}(w)| < |u-w| \int_0^\xi x^s |g(x)|\, dx +$$

$$+ 2 \int_\xi^\infty x^{s-1} |g(x)|\, dx. \quad (7.22)$$

Page 431.

This inequality is correct with everyone ξ, but when $\xi = \frac{2}{|u-w|}$ the

value of the right side of the inequality will be smallest. After dividing into $|u-w|$ and using designation (7.18), we obtain

$$|\Delta(u, w)| \leqslant \int_0^{\xi} x^s |g(x)|\, dx + \xi \int_{\xi}^{\infty} x^{s-1} |g(x)|\, dx, \qquad (7.23)$$

whence after the simple conversion of right side is obtained formula (7.17).

It is now easy to find for g(x) the condition, sufficient for absolute integrability Δ(u, w) and, consequently, also F(w).

Integrating inequality (7.17) and assuming w>u, we obtain

$$\int_u^{\infty} |\Delta(u, v)|\, dv \leqslant 2 \int_{\xi}^{\infty} \frac{dx_1}{x_1^2} \int_0^{x_1} h(x)\, dx \qquad (7.24)$$

or

$$\int_u^{\infty} |\Delta(u, v)|\, dv \leqslant \frac{2}{\xi} \int_0^{\xi} h(x)\, dx + 2 \int_{\xi}^{\infty} \frac{h(x)}{x}\, dx, \qquad (7.25)$$

where ξ has previous value (7.18).

Since by hypothesis integral (7.16) descends, then function h(x) approaches at infinity zero. Consequently, with $\xi \to \infty$ vanishes the first member in right side (7.25). Thus, sufficient condition for the absolute integrability Δ(u, v) is the convergence of the integral

$$h_1(\xi) = \int_{\xi}^{\infty} \frac{h(x)}{x}\, dx. \qquad (7.26)$$

For satisfaction of this condition it is still insufficient so that the integral for h(x) would descend; it is necessary that it would vanish not too slowly.

Integral $h_1(\xi)$ it is not difficult to express directly through g(x). Substituting (7.16) in (7.26) and varying the order of integration for Dirichlets's formula, we will obtain

$$h_1(\xi) = \int_{\xi}^{\infty} (\lg x - \lg \xi) x^{s-1} |g(x)| dx. \qquad (7.27)$$

Page 432.

Hence it is apparent that in the case of real roots all conditions will be satisfied and function F(w) will be absolutely integrated in any final gap/interval under the only further condition for g(x), namely under the condition of the convergence of the integral

$$g_3(\xi) = \int_{\xi}^{\infty} (\lg x) x^{s-1} |g(x)| dx, \qquad (7.28)$$

where $s \geq 1$ - greatest multiplicity of real roots.

If integral (7.28) descends, then function h(x) will decrease more rapidly than is inversely proportional lgx. If it decreases so rapidly that product $x^{\alpha} h(x)$ remains limited, then, as can be seen from (7.17), function $G_1^{(s-1)}(w)$ will satisfy Lipshitz condition

$$|G_1^{(s-1)}(w) - G_1^{(s-1)}(u)| < M |u - w|^{\alpha}. \qquad (7.29)$$

It is easy to show also that then will occur the inequality

$$|D_s(w)| < \frac{M}{(s-1)!} |w - w_m|^{\sigma-1} \qquad (7.30)$$

with the same value of M.

We explained that with the convergence of integral (7.28) function F(w) will be absolutely integrated in any final gap/interval. Let us consider now behavior F(w) at infinity.

For research F(w) we will use expression (6.41). For functions $\psi_1(w)$ and $H_1(w)$ occur the same estimations, as in the absence of real roots. Designating through p_0 coefficient with the old degree of w in polynomial $P_{n-1}(\bar{w})$, we will have with $|w| \to \infty$

$$F(w) = \psi_1(w)\left[H_1(w) + \frac{p_0}{w}\right] + 0\left(\frac{1}{|w|^2}\right). \qquad (7.31)$$

Hence it is apparent that function F(w) will be at infinity of the same character as in the absence of real roots, and to it will be applicable formula (4.20) (with replacement of iC on iC+p_0). The same conclusion relates also to Φ (w).

Comparing this result with recently proved absolute integrability F(w) in any final gap/interval, we come to the conclusion that the reasonings of paragraphs 4 and 5 require no changes. Thus, and for the case of real roots it is possible to take for granted that with those done relative to g(x) and k(x) the

assumptions the integral, expressions f(x) through F(w), exists, and that the determined by it function f(x) satisfies the integral equation proposed.

Page 433.

8. Conclusion/output of the conditions of orthogonality.

We will continue the research here initiated in paragraph 6 of the case of real roots and let us study in greater detail than those conditions of orthogonality by which must satisfy the assigned function g(x).

Let us assume that all chosen roots $w_1, w_2, \ldots, w_n$ are real. As it was already indicated in paragraph 7, this always can be achieved/reached by the contraction of the band in question in plane w. If v_0 - alleged part of the complex root 1-K(w)nearest to the real axis, or the nearest singular point of this function, then to sufficiently take

$$c' < v_0 \tag{8.01}$$

and to examine the band

$$-c' \leqslant \mathrm{Im}(w) \leqslant c'. \tag{8.02}$$

Let us designate through S(w) the polynomial of degree 2n, comprised

of factors $w^2 - w_m^2$:

$$S(w) = (w^2 - w_1^2)\dots(w^2 - w_n^2). \qquad (8.03)$$

Expression (6.42) for function F(w) can be written in the form

$$F(w) = \frac{\psi_{n1}(w)}{S(w)}\left[(w - ib)^n H_1(w) + P_{n-1}(w)\right]. \qquad (8.04)$$

Let us examine in more detail the relationship/ratio by which must satisfy function H_1(w). Let us lead at plane w the locked duct/contour G, which surrounds all points w=+- w_m and which lies within band (8.02) or on its boundary. [It is possible, for example, to take rectangular duct/contour of two segments of lines, components band edge (8.02), and two segments, parallel to imaginary axis]. Let us assume

$$R(w) = (w - ib)^n H_1(w) \qquad (8.05)$$

and let us designate through R_{an} (w) any analytic function, holomorphic within G and which takes at points w=+- w_m the same values, as R(w). If root w_m — is multiple, multiplicity s, then at point w_m must coincide derivatives of R_{an} (w) and from R(w) of order to s-1 inclusively. [As R_{an} (w) it is possible to take polynomial sufficiently high degree, and if function itself R(w) is holomorphic within G, then it is possible simply to assume $R_{an}(w) = R(w)$.]

Page 434.

In formula (8.04) polynomial $P_{n-1}(w)$ must be determined from the condition

$$R(w) + P_{n-1}(w) = 0 \ \text{при}^{(1)} \ w = \pm w_m \qquad (8.06)$$

Key: (1). with.

or, which is the same thing, from the condition

$$R_{an}(w) + P_{n-1}(w) = 0 \ \text{при}^{(1)} \ w = \pm w_m. \qquad (8.07)$$

Key: (1). with.

moreover the degree of polynomial $P_{n-1}(w)$ must be not above n-1.

If this determination is possible, then all integrals

$$I_r = \frac{1}{2\pi i}\int_{\Gamma} w^r \frac{R_{an}(w) + P_{n-1}(w)}{S(w)} dw, \qquad (8.08)$$

where r - positive integer number, they must become zero. However, since the degree of polynomial $P_{n-1}(w)$ exists n-1, then for r=0, 1, ..., n-1 will be individually

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{w^r P_{n-1}(w)}{S(w)} dw = 0, \qquad (8.09)$$

since here under the integral stands the rational fraction in which the degree of denominator, at least, per two units higher than degree of numerator.

Therefore the conditions for existence of polynomial $P_{n-1}(w)$ take the form

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{w^r R_{an}(w)}{S(w)} dw = 0 \quad (r = 0, 1, \ldots, n-1). \qquad (8.10)$$

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Here the integral confronting is equal to the sum of deductions at points w=+- w_m. After agreeing to count

$$w_{-m} = -w_m, \qquad (8.11)$$

we can rewrite conditions (8.10) in the form

$$\sum_{m=-n}^{+n}{}' \frac{R_{an}(w_n)}{S'(w_m)} w_m^r = 0, \qquad (8.12)$$

if roots w_m simple, and into somewhat more complicated form, if they multiple (prime in the sign of sum it means that the term, for which m=0, must be omitted). But at points w=w_m analytic function R_{an} (w) takes the same values, as R(w). Substituting for R(w) expression (8.05), we will have

$$\sum_{m=-n}^{+n} \frac{(w_m - ih)^n}{S'(w_m)} w_m^r H_1(w_m) = 0 \quad (r = 0, 1, \ldots, n-1). \ (8.13)$$

We obtained in the explicit form those algebraic relationships/ratios by which must satisfy (in the case of simple roots) function H_1(w). Let us pass from them to the relationships/ratios for G_2(w). According to determination of H_1(w)

$$\psi_2(w)\, G_1(w) = H_1(w) + H_2(w). \qquad (8.14)$$

Page 435.

Therefore relationships/ratios (8.13) are equivalent to the following:

$$\sum_{m=-n}^{+n}{}' \frac{(w_m - ib)^n}{S'(w_m)} w_m^r \psi_2(w_m) G_1(w_m) =$$

$$= \sum_{m=-n}^{+n}{}' \frac{(w_m - ib)^n}{S'(w_m)} w_m^r H_2(w_m). \qquad (8.15)$$

Let us designate here the right side through A and we convert it. Using the fact that function $H_2(w)$ is holomorphic in band (8.02), and also, therefore, within the duct/contour G, we can represent A in the form of the integral

$$A = \frac{1}{2\pi i} \int_{\Gamma} \frac{(w - ib)^n w^r H_2(w)}{S(w)} dw. \qquad (8.16)$$

Duct/contour G can be replaced by two with straight lines, parallel real axes; then value A can be represented in the form of the difference

$$A = A_1 - A_2 \qquad (8.17)$$

in two absolutely convergent integrals

$$A_1 = \frac{1}{2\pi i} \int_{-ic'-\infty}^{-ic'+\infty} \frac{(w - ib)^n w^r H_2(w)}{S(w)} dw, \qquad (8.18)$$

$$A_2 = \frac{1}{2\pi i} \int_{ic'-\infty}^{ic'+\infty} \frac{(w - ib)^n w^r H_2(w)}{S(w)} dw. \qquad (8.19)$$

The first of these integrals is equal to zero, since integrand in it is holomorphic in the lower half-plane, and at infinity it

vanishes as $|w|^{r-n-1}$, i.e. at least conversely $|w|^2$. However, in the second integral we again can express with the help of (8.14) $H_2(w)$ through $H_1(w)$ and through $\psi_2(w)G_1(w)$. It is easy to see that the integral, obtained from A_2 by replacement of $H_2(w)$, is equal to zero. Therefore

$$A_2 = \frac{1}{2\pi i} \int_{ic'-\infty}^{ic'+\infty} \frac{(w-ib)^n w^r \psi_2(w)\, G_1(w)}{S(w)}\, dw, \tag{8.20}$$

whereas

$$A_1 = 0. \tag{8.21}$$

Page 436.

Since $A=-A_2$ is right side (8.15), the condition (8.13) can be formulated as follows:

$$\sum_{m=-n}^{+n}{}' \frac{(w_m-ib)^n}{S'(w_m)} w_m^r \psi_2(w_m)\, G_1(w_m) =$$

$$= -\frac{1}{2\pi i} \int_{ic'-\infty}^{ic'+\infty} \frac{(w-ib)^n}{S(w)} w^r \psi_2(w)\, G_1(w)\, dw. \tag{8.22}$$

The obtained formula can be written into somewhat simpler form, if we assume according to (6.07)

$$\psi_{n2}(w) = (w-ib)^n \psi_2(w). \tag{8.23}$$

Then

$$\sum_{m=-n}^{+n} \frac{\psi_{n2}(w_m)}{S'(w_m)} w_m^r G_1(w\ ) =$$

$$= -\frac{1}{2\pi i} \int_{ic'-\infty}^{ic'+\infty} \frac{\psi_{n2}(w)}{S(w)} w^r G_1(w)\,dw \quad (r = 0, 1, \ldots, n-1). \quad (8.24)$$

Let us recall that ψ_{n2} (w), in contrast to ψ_2 (w), does not depend on the selection of number b. According to the estimation, ob aained from (4.12) by replacement ψ_1 on ψ_2, we have in half-plane Im(w)<c' with the sufficiently large | w |

$$|\psi_{n2}(w) - w^n| < M|w|^{n-1} \lg|w|. \qquad (8.25)$$

Hence it is apparent that the integral in (8.24) will be that absolutely converging.

In formula (8.24) it is possible to substitute for G_1(w) the integral expression

$$G_1(w) = \int_0^\infty e^{ixw} g(x)\,dx. \qquad (8.26)$$

Using (8.25), it is easy to demonstrate the legitimacy of change in right side (8.24) of the order of integration, so that the first integration can be produced on w.

Page 437.

After assuming

$$\beta_r(x)=\frac{i^{r-n}}{2\pi}\int\limits_{ic'-\infty}^{ic'+\infty} e^{ixw}\frac{\psi_{ns}(w)}{S(w)}w^r\,dw, \qquad (8.27)$$

we can write right side (8.24) in the form

$$A=-i^{n-r-1}\int\limits_0^\infty g(x)\beta_r(x)\,dx. \qquad (8.28)$$

In the analogous form it is possible to represent left side (8.24). After assuming

$$\alpha_r(x)=i^{r-n+1}\sum_{m=-n}^{+n}{}'\frac{\psi_{ns}(w_m)}{S'(w_m)}w_m^r e^{ixw_m}, \qquad (8.29)$$

we will obtain for left side (8.24) the expression

$$\sum_{m=-n}^{+n}{}'\frac{\psi_{ns}(w_m)}{S'(w_m)}w_m^r G_1(w_m)=i^{n-r-1}\int\limits_0^\infty g(x)\alpha_r(x)\,dx, \qquad (8.30)$$

which by force of (8.15) must be equal by A. Equalizing (8.28) and (8.30), we will obtain conditions for g(x) in the form

$$\int\limits_0^\infty g(x)\gamma_r(x)\,dx=0 \quad (r=0,1,\ldots,n-1), \qquad (8.31)$$

where

$$\gamma_r(x)=\alpha_r(x)+\beta_r(x). \qquad (8.32)$$

Thus, purely algebraic condition (8.13) for $H_1(w)$ proved to be equivalent to the condition of orthogonality (8.31) for g(x).

9. Properties of the functions, entering the conditions of orthogonality.

Let us study in greater detail than the functions, entering the conditions of orthogonality.

Value α_r (x) can be represented in the form of the contour integral

$$\alpha_r(x)=\frac{i^{r-n}}{2\pi}\int_{\Gamma}e^{ixw}\frac{\psi_{n_2}(w)\,w^r}{S(w)}\,dw. \qquad (9.01)$$

This is representation correctly also in the case of multiple roots.

Substituting duct/contour G by two parallel lines, we can represent integral (9.01) in the form of a difference in two integrals, similarly how this is done in formula (8.17), where integral A is given in the form of difference A_1-A_2. One of them it will be exactly equal to integral (8.27) for β_r (x). Composing sum (8.32), we will obtain

$$\gamma_r(x)=\frac{i^{r-n}}{2\pi}\int_{-ic'-\infty}^{-ic'+\infty}e^{ixw}\frac{\psi_{n_2}(w)}{S(w)}\,w^r\,dw. \qquad (9.02)$$

Page 438.

Function γ_r (x) will be the derivative of the r order of $\gamma_0(x)$:

$$\gamma_r(x)=\frac{d^r}{dx^r}\gamma_0(x). \qquad (9.03)$$

Analogous relationships/ratios there exists for α_r (x) and for $\beta_r(x)$.

Since factor with the exponential function under integral (9.02) is the function, holomorphic in the lower half-plane, moreover it vanishes at infinity at least conversely $|w|$, then at the negative values of x integral (9.02) is equal to zero:

$$\gamma_r(x) = 0 \quad \overset{(1)}{\text{при}}\ x < 0. \tag{9.04}$$

Key: (1). with.

All functions γ_r (x), except γ_{n-1} (x), are continuous. In particular,

$$\gamma_r(0) = 0 \quad (r = 0, 1, \ldots, n-2). \tag{9.05}$$

However, function γ_{n-1} (x) undergoes with x=0 the interruption/discontinuity:

$$\gamma_{n-1}(+0) = 1, \quad \gamma_{n-1}(-0) = 0. \tag{9.06}$$

Let us consider behavior γ_r (x) with large positive x. It is easy to see that with the unlimited increase x functions β_r (x) vanish according to the exponential law. In fact, from formula (8.27) it is evident that it is possible to assume

$$\beta_r(x) = e^{-c'x}\beta_r^*(x), \tag{9.07}$$

where β_r^* (x) remains limited with the infinite increase x.

Coefficient c' in the index satisfies inequality (8.01), but it can be undertaken as to close ones as desired to v_*.

As far as value $\alpha_r(x)$, is concerned, for it we have explicit expression (8.29). It is directly applicable in the case of simple roots. None of the factors $\psi_{n2}(w_m)$ in it is equal to zero. Value $\alpha_r(x)$ will be the oscillating function from x, in which will be represented all "frequencies" w_m. This function although remains limited, it approaches at infinity not what limit. If there are multiple roots, then expression for α_r (x) can be obtained either from (8.29), by considering multiple roots as the limit of close roots, or it is direct from integral (9.01). This expression will contain the products of exponential factors e^{ixw_m} to the polynomials from x degree $s_m - 1$ (s_m — the multiplicity of root w_m). If s - greatest of the numbers s_m, the relation $\alpha_r(x) : x^{s-1}$ remains limited and it will approach not what limit.

Not one of the functions $\alpha_r(x)$,, and also any linear combination of these functions they will at infinity vanish.

Page 439.

Hence, in particular, it follows that functions α_r (x) are linearly independent. The formal proof of the linear independence of these functions is simplest to conduct on the basis of the following lemma.

Lemma IV. Let the function $\varphi(w)$ be holomorphic within the locked duct/contour G, with exception, perhaps, poles, and let it be it is given, which

$$\frac{1}{2\pi i}\int_{\Gamma} e^{ixw}\varphi(w)\,dw = 0 \qquad (9.08)$$

is identical relative to x. Then $\varphi(w)$ does not have poles within G.

Proof. Let us isolate principal part in $\varphi(w)$:

$$\varphi(w) = \sum_{p,\,n}\frac{A(p,\,n)}{(w-p)^{n+1}} + \varphi_1(w), \qquad (9.09)$$

where $\varphi_1(w)$ it is holomorphic within G. Then

$$\frac{1}{2\pi i}\int_{\Gamma} e^{ixw}\varphi(w)\,dw = \sum_{p,\,n} A(p,\,n)(ix)^n e^{ipx}. \qquad (9.10)$$

This expression can become identically zero only in such a case, when all coefficients A(p, n) are equal to zero, i.e., if $\varphi(w)$ is led to the holomorphic function $\varphi_1(w)$, QED.

Passing to the proof of linear independence $\alpha_r(x)$, let us assume reverse/inverse, i.e., let us assume that between these functions there is a relationship/ratio of the form

$$A_0\alpha_0(x) + A_1\alpha_1(x) + \cdots + A_{n-1}\alpha_{n-1}(x) = 0. \qquad (9.11)$$

Substituting here expressions for $\alpha_r(x)$ in the form of contour integrals (9.01) and designating through P(w) a polynomial

$$P(w) = A_0 + A_1(iw) + A_2(iw)^2 + \cdots + A_{n-1}(iw)^{n-1}, \quad (9.12)$$

we would have

$$\int_{\Gamma} e^{ixw} \frac{\psi_{n2}(w)\, P(w)}{S(w)}\, dw = 0 \quad (9.13)$$

with everyone x.

Page 440.

Hence on the basis of lemma IV we consist that the function

$$\varphi(w) = \frac{\psi_{n2}(w)\, P(w)}{S(w)} \quad (9.14)$$

is holomorphic within G, but since $\psi_{n2}(w)$ is there holomorphic and does not have zero, then holomorphic must be the function

$$\frac{\varphi(w)}{\psi_{n2}(w)} = \frac{P(w)}{S(w)}, \quad (9.15)$$

which is impossible, if only polynomial (9.12) is not equal to zero [since P(w) is identically polynomial of degree n-1, but S(w) - polynomial of degree 2n]. Consequently, there is no relationships/ratios of form (9.11) with the different from zero ones coefficients A_r, and function $\alpha_r(x)$ they are linearly independent.

Since functions $\alpha_r(x)$ do not vanish at infinity, and functions $\beta_r(x)$ decrease there according to the exponential law, the behavior of functions $\gamma_r(x)$ with $x \to +\infty$ is determined by the behavior of the functions $\alpha_r(x)$. Whence it follows first, that function $\gamma_r(x)$ they are linearly independent and, in the second place, that none of them and

their any linear combination can at infinity vanish.

In conclusion let us note that since relation $\gamma_r(x):x^{s-1}$ remains limited, then all integrals (8.31), which express the conditions of orthogonality, will be, under our assumptions relative to g(x), which absolutely converge [see formula (7.28)].

10. Solution of homogeneous equation.

Let us demonstrate that the entering the conditions orthogonalities of function $\gamma_r(x)$ represent the solutions of the homogeneous equation

$$\gamma_r(x)=\int_0^\infty k(x-y)\,\gamma_r(y)\,dy. \qquad (10.01)$$

for this let us consider the integral

$$\int_{-\infty}^{+\infty} k(x-y)\,e^{iwy}\,dy=e^{ixw}K(w), \qquad (10.02)$$

which will be that absolutely converging, until w lies/rests at band (4.09) and those in band (8.02). Let us multiply it on

$$\frac{i^{r-n}}{2\pi}\frac{\psi_{n2}(w)}{S(w)}w^r\,dw$$

and let us integrate in the limits from -ic'-N to -ic'+N.

Page 441.

As a result of the absolute convergence of integral (10.02) we can

change the order of integration and we will obtain

$$\frac{i^{r-n}}{2\pi}\int_{-ic'-N}^{-ic'+N} e^{ixw} K(w) \frac{\psi_{n2}(w)}{S(w)} w^r \, dw =$$

$$= \int_{-\infty}^{+\infty} k(x-y)\,\gamma_{rN}(y)\,dy, \qquad (10.03)$$

where it is placed

$$\gamma_{rN}(y) = \frac{i^{r-n}}{2\pi}\int_{-ic'-N}^{-ic'+N} e^{iwy} \frac{\psi_{n2}(w)}{S(w)} w^r \, dw. \qquad (10.04)$$

Discussing as in paragraph 5, it is easy to demonstrate that

$$|e^{-c'y}\gamma_{rN}(y)| < L, \qquad (10.05)$$

where L depends neither on N nor on y.

On the other hand, the function

$$e^{c'y}k(x-y) \qquad (10.06)$$

will be absolutely integrated. Therefore in right side (10.03) it is possible to pass to the limit under the integral sign. As a result we will obtain

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty} k(x-y)\,\gamma_{rN}(y)\,dy = \int_{-\infty}^{+\infty} k(x-y)\,\gamma_r(y)\,dy. \qquad (10.07)$$

The limit of left side (10.03) is equal to

$$\frac{i^{r-n}}{2\pi}\int\limits_{-ic'-\infty}^{-ic'+\infty} e^{ixw}K(w)\frac{\psi_{n2}(w)}{S(w)}w^r\,dw =$$

$$= \gamma_r(x) - \frac{i^{r-n}}{2\pi}\int\limits_{-ic'-\infty}^{-ic'+\infty} e^{ixw}[1-K(w)]\,\frac{\psi_{n2}(w)}{S(w)}w^r\,dw. \quad (10.08)$$

Using the equality

$$[1-K(w)]\,\frac{\psi_{n2}(w)}{S(w)} = \frac{1}{\psi_{n1}(w)}, \quad (10.09)$$

resulting from (6.05), easily we are convinced, that at the positive values of x an integral in right side (10.08), equals zero.

Equating (10.07) and (10.08), we will have with x>0

$$\int\limits_{-\infty}^{+\infty} k(x-y)\,\gamma_r(y) = \gamma_r(x) \quad (x>0), \quad (10.10)$$

and as a result of (9.04)

$$\int\limits_0^{\infty} k(x-y)\,\gamma_r(y)\,dy = \gamma_r(x). \quad (10.11)$$

Page 442.

We obtained the result which could be foreseen: those functions to which must be orthogonal the absolute term g(x) of nonhomogeneous equation, they are the solutions of homogeneous equation.

Let us demonstrate that functions $\gamma_r(x)$ will be the unique solutions of homogeneous equation, which satisfy the inequality

$$|\gamma_r(x)| < Le^{v'x} \quad (0 < v' < v_0), \quad (10.12)$$

i.e. increasing slower than according to the exponential law.

For the proof let us consider the nonhomogeneous integral equation

$$f(x)=g(x)+\int_0^{\infty}k(x-y)f(y)\,dy \qquad (10.13)$$

and let us select in it the absolute term g(x) so that its solution f(x) rapidly would decrease. For this it suffices to require so that g(x) would satisfy the same conditions as kernel k(x); we let us assume that not only g(x), but also the function

$$g_1(x)=e^{cx}g(x), \qquad (10.14)$$

where c has the same value, as in (4.01) it is absolutely integrated and has the bounded variation in the infinite gap/interval.

It is not difficult to see that function $G_1(w)$ will be then holomorphic in the same band, as K(w) (and, furthermore, of course, in the upper half-plane). From expression (7.02) for F(w) it is evident that the unique singular points F(w) within band (4.09) can be the roots 1-K(w), which lie on the real axis or lower than it; these roots will be poles F(w). But if function g(x) satisfies the conditions of orthogonality (8.31), then on the real axis function F(w) poles does not have, and the nearest to the real axis singular point F(w) will have alleged part - v_0. Thus, F(w) will be holomorphic in the half-plane

$$\mathrm{Im}\,(w) > -v_0. \qquad (10.15)$$

Let v - number, which satisfies the inequality

$$0<v<v_0 \qquad (10.16)$$

(v it can be as to close ones as desired to v_0). Let us write integral for f(x) in the form

$$f(x)=\frac{e^{-vx}}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}F(u-iv)\,du. \qquad (10.17)$$

Page 443.

Hence it is apparent that it is possible to assume

$$f(x)=e^{-vx}f_1(x). \qquad (10.18)$$

where function $f_1(x)$ remains limited with the infinite increase x:

$$|f_1(x)|<L_1. \qquad (10.19)$$

Thus, during the proper selection of absolute term there are solutions of nonhomogeneous equation, which rapidly decrease at infinity.

Let us assume now that, besides the functions

$$\gamma_0(x),\ \gamma_1(x),\ \ldots,\ \gamma_{n-1}(x), \qquad (10.20)$$

obtained above there is one additional solution of the homogeneous equation

$$\gamma(x)=\int_0^{\infty} k(x-y)\,\gamma(y)\,dy. \qquad (10.21)$$

which grows more slowly than $e^{v'x}$, so that

$$|\gamma(x)| < e^{v'x}L \qquad (0 < v' < v_0). \qquad (10.22)$$

where L - constant.

Let us fit in formula (10.16) value v in such a way that it would be

$$v' < v < v_0. \qquad (10.23)$$

Let us multiply both parts (10.13) on $\gamma(x)$ and will integrate over x from 0 to ∞. Double integral in the right side will be that absolutely converging, since it will be less than

$$LL_1\int_0^\infty\int_0^\infty e^{v'x-vy}|k(x-y)|\,dx\,dy <$$
$$< LL_1\int_0^\infty e^{-(v-v')y}\,dy\int_{-\infty}^{+\infty} e^{v'(x-y)}|k(x-y)|\,dx. \qquad (10.24)$$

Simple integrals will also be, obviously, those absolutely converging. After changing in the double integral the order of integration, we will obtain

$$\int_0^\infty f(y)\left\{\gamma(y)-\int_0^\infty k(x-y)\,\gamma(x)\,dx\right\}dy=\int_0^\infty g(x)\,\gamma(x)\,dx. \qquad (10.25)$$

Since $\gamma(x)$ satisfies homogeneous equation, then as the necessary condition we will obtain one additional condition of the orthogonality:

$$\int_0^\infty g(x)\,\gamma(x)\,dx = 0. \qquad (10.26)$$

Page 444.

But we already proved that the previous conditions of orthogonality it is sufficient for the existence of solution. Therefore new condition (10.26) must be the corollary of previous conditions (8.31).

Hence it follows that $\gamma(x)$ will be the linear combination of functions $\gamma_m(x)\ (m = 0, 1, \ldots, n-1)$. In fact, let us take as g(x) function [1]

$$g(x) = e^{-2\nu_0 x} \{a_0\bar{\gamma}_0(x) + a_1\bar{\gamma}_1(x) + \cdots + a_{n-1}\bar{\gamma}_{n-1}(x) - \bar{\gamma}(x)\} \quad (10.27)$$

and let us fit in it coefficients of $a_0, a_1, \ldots, a_{n-1}$, so that g(x) would be orthogonal to all functions (10.20).

FOOTNOTE. The feature above the letter designates the complex conjugate value. ENDFOOTNOTE.

This is always possible, since as a result of the linear independence of these functions the determinant of the system of linear equations for $a_0, a_1, \ldots, a_{n-1}$, (Gram's determinant) is surely different from

zero. But if g(x) is orthogonal to functions (10.20), then on that proved it is orthogonal both to the function $\gamma(x)$and, consequently, also and tc the function

$$\varepsilon(x)=\bar{a}_0\gamma_0(x)+\cdots+\bar{a}_{n-1}\gamma_{n-1}(x)-\gamma(x), \qquad (10.28)$$

so that

$$\int_0^\infty e^{-2\sigma_0 x}\,|\,\varepsilon(x)\,|^2\,dx=0. \qquad (10.29)$$

This equality can occur only in such a case, when

$$\varepsilon(x)=0 \qquad (10.30)$$

it is identical relative to x, that also proves the linear dependence $\gamma(x)$ on $\gamma_0(x),\ldots,\gamma_{n-1}(x)$.

Thus, we demonstrated that the homogeneous equation has exactly n of solutions (10.20), which satisfy condition (10.22), and any other solutions, besides those found, do not exist.

The result obtained by us makes it possible very simply to solve a question also about the uniqueness of that obtained by us more equal the solution of nonhomogeneous equation.

In fact, if there would be two solutions of the nonhomogeneous equation which both would be approached zero at infinity, then their difference would represent such solution of homogeneous equation which at infinity also would approach zero. But such solution (not

equal to zero it is identical) there does not exist, since the very general solution of the homogeneous equations is the linear combination of functions (10.20), and no linear combination these of functions becomes at infinity zero.

In conclusion let us make one observation about those solutions of the homogeneous equation which do not satisfy condition (10.22).

Page 445.

Such solutions $\gamma(x)$, if they exist, can be represented in the form of integral (9.02), if we include/connect in a number of roots S(w) also complex roots. The asymptotic form of these solutions at infinity will be given by expression (8.29) or, for the multiple roots, by its maximum form where already among values w_m there will be complex with the negative alleged part.

If we subordinate function g(x) to the same conditions as k(x) [see (10.14)], then will make sense the integrals of the form

$$I=\int_0^\infty g(x)\,\gamma(x)\,dx, \qquad (10.31)$$

in spite of increase γ(x). Let c' - certain positive number, smaller c. It is possible to require so that integrals (10.31) would be equal to zero for all solutions $\gamma(x)$ homogeneous equation, which increase

not more rapid $e^{c'x}$. Then function F (w) is holomorphic in the entire half-plane Im(w)>-c', and f(x) will decrease at least as $e^{-c''x}$ (c'' — any positive number less than c').

11. The conditions for existence of solution, to which is applicable Fourier's formula.

In the beginning of our research we assumed that solution f(x) of our integral equation is such that to it was applicable Fourier's formula. Subsequently we were freed from this assumption and demonstrated existence and uniqueness of solution independent of it. Nevertheless it is interesting to find conditions, with which the initial assumption occurs, since the solution to which is applicable Fourier's formula, it will be in any case unique solution with this property.

The applicability of Fourier's formula to f(x) indicates the same as the applicability of this formula to F(w) for real w. From the fundamental functional equation we have

$$F(w) - \ddot{K}(w) F(w) = G_1(w) + \ddot{\Phi}(w). \qquad (11.01)$$

To function $G_1(w)$ Fourier's formula is applicable according to property g(x). Function $\Phi(w)$ is holomorphic in the band, which switches on real axis; therefore in order to be convinced of the applicability to it of Fourier's formula, it suffices to recall, that

with large $|w|$ is correct expression (4.25), and to check that derivative $\Phi'(w)$ is absolutely integrated in entire infinite gap/interval. During this checking we stop will not be.

Page 446.

Thus, to right side (11.01) Fourier's formula surely the applicability of this formula to F(w) will be proved, if it will be proved for K(w)F(w). But function K(w)F(w) is absolutely integrated in the infinite gap/interval, and us it suffices to show that the derivative of it is absolutely integrated in any final gap/interval.

We have

$$K(w)F(w)=\frac{K(w)}{1-K(w)}[G_1(w)+\Phi(w)]. \tag{11.02}$$

If denominator in (11.02) does not have real roots, then for the absolute integrability of derivative of (11.02) it is sufficient so that the derivative $G'_1(w)$ would be limited. For this it suffices to assume the absolute integrability $x\cdot g(x)$ in the infinite gap/interval.

If denominator has real roots, then near each root $w=w_m$ it is possible to use representation F(w) in the form (7.03). Discussing as in paragraph 7, and using the expression

$$-\frac{dD_s}{dw}=\int_0^1\frac{G_1^{(s)}(wt+w_m(1-t))-G_1^{(s)}(w_m)}{w-w_m}\frac{d}{dt}\frac{t(1-t)^{s-1}}{(s-1)!}dt, \tag{11.03}$$

and also by the formula

$$s\int_0^1 \left|\frac{d}{dt}t(1-t)^{s-1}\right| dt \leqslant 1, \qquad (11.04)$$

it is easy to establish/install for $w > w_m$ the inequality

$$\int_{w_m}^{w} \left|\frac{dD_s}{dw}\right| dw \leqslant \frac{1}{s!}\int_{w_m}^{w} \left|\frac{G_1^{(s)}(u) - G_1^{(s)}(w_m)}{u - w_m}\right| du, \qquad (11.05)$$

the analogous to inequality (7.05). Using results of paragraph 7, we consist hence that sufficient condition for absolute integrability $D_s'(w)$ near each root will be the convergence of the integral

$$g_4(\xi) = \int_{\xi}^{\infty} (\lg x)\, x^s\, |g(x)|\, dx, \qquad (11.06)$$

where s - greatest multiplicity of real roots.

But if $D_s'(w)$ is absolutely integrated near each root, then the derivative F' (w) will be absolutely integrated in any final gap/interval and all conditions for the applicability to functions K(w) F(w) and F(w) of Fourier's formula will be satisfied.

Page 447.

Thus when s - greatest multiplicity of real roots, then the condition for existence of solution f(x), to which it is applicable

Fourier's formula, they will be in the accuracy the same, what were for multiplicity s+1 of the condition for existence and uniqueness of the solution, which vanishes at infinity, namely:

1° absolute integrability and the limitedness of variation g(x) in the infinite gap/interval and

2° absolute integrability $(\lg x)\, x^s g(x)$ in the infinite gap/interval [1].

FOOTNOTE [1]. The conditions of orthogonality for g(x) we do not mention, since they are always assumed to be those carried out. ENDFOOTNOTE.

If real roots are absent, then for the existence of solution, to which is applicable Fourier's formula, instead of condition of 2° it suffices to require the absolute integrability xg(x) in the final gap/interval.

12. Examples.

Before passing for particular examples, let us consider that frequently encountered in the physical tasks case, when the absolute

term of integral equation is the exponential function

$$g(x) = e^{ipx} \qquad (\operatorname{Im}(p) > 0). \qquad (12.01)$$

Then according to formula (1.11)

$$G_1(w) = \frac{i}{w+p}. \qquad (12.02)$$

Expansion (3.14) can be carried out in the explicit form, precisely,:

$$\psi_2(w)\frac{i}{w+p} = \frac{i\psi_1(p)}{w+p} + i\,\frac{\psi_2(w)-\psi_2(-p)}{w+p}. \qquad (12.03)$$

Here we used relationship/ratio (3.10). It is obvious that in formula (12.03) the first term to the right exists $H_1(w)$, and the second term exists $H_2(w)$. According to formula (3.16) we have

$$F(w) = i\,\frac{\psi_1(w)\,\psi_1(p)}{w+p}, \qquad (12.04)$$

so that function F(w) is symmetrical relative to argument w and parameter p.

Thus, the solution of integral equation with absolute term (12.01) takes the form

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{-ixw}\,\frac{\psi_1(w)\,\psi_1(p)}{w+p}\,dw. \qquad (12.05)$$

Page 448.

If assumes here x=0, then according to Fourier's formula we will obtain the half-sum

$$f(0) = \frac{1}{2}\,[f(+0) + f(-0)], \qquad (12.06)$$

and since f(-0)=0, then

$$f(+0) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\psi_1(w)\,\psi_1(p)}{w+p}\,dw. \qquad (12.07)$$

Integration for the real axis can be replaced in this case with integration for the semicircumference of the infinitely increasing radius, which lies at the upper half-plane. Since at infinity $\psi_1(w)$ it is converted into unity, then integral easily is computed, and we obtain

$$f(+0) = \psi_1(p). \qquad (12.08)$$

On the other hand, assuming/setting in the initial integral equation x=0, we find

$$f(+0) = 1 + \int_0^\infty k(y) f(y)\, dy. \qquad (12.09)$$

In the left side it is possible to substitute (12.08), and integral in the right side to replace with its value from (5.17), where it is necessary, in turn, to substitute for F(w) expression (12.04). As a result we will obtain

$$\psi_1(p) = 1 + \frac{i}{2\pi} \int_{-\infty}^{+\infty} K(w) \frac{\psi_1(w)\,\psi_1(p)}{w+p}\, dw. \qquad (12.10)$$

This relationship/ratio can be considered as nonlinear functional equation for $\psi_1(w)$. Solution it to us is already known: it is given according to (3.06) and (3.08) by the use/application of lemma I to function (3.04), in other words, by expansion 1 - K(w) to the factors according to formula (3.09). After substituting in (12.10) the expression for K(w) from (3.09) and using the general/common/total

properties of function $\psi_1(w)$. easy of this to be convinced and directly.

As the simplest example let us consider the equation

$$f(x)=g(x)+\lambda\int_0^\infty e^{-|x-y|}f(y)\,dy, \qquad (12.11)$$

in which for simplicity we will consider it λ real. Here kernel k(x) is equal

$$k(x)=\lambda e^{-|x|}. \qquad (12.12)$$

Page 449.

The converted kernel is computed from general formula (1.11) and it will be equally

$$K(w)=\frac{2\lambda}{1+w^2}. \qquad (12.13)$$

In this simple case value K(w) to the conversion already takes form (6.20). Roots 1-K(w) are given by formula $w=\pm\sqrt{2\lambda-1}$. They will be complex with 2λ<1 and they are real with 2λ>1.

Case 1 (2λ<1). Let us assume

$$2\lambda=1-\mu^2 \quad (\mu>0). \qquad (12.14)$$

so that

$$K(w)=\frac{1-\mu^2}{1+w^2}, \quad 1-K(w)=\frac{w^2+\mu^2}{w^2+1}. \qquad (12.15)$$

Applying the reasonings of paragraph 3 and factoring 1-K(w), we must in formula (3.09) assume

$$\psi_1(w)=\frac{w+i}{w+i\mu}, \quad \psi_2(w)=\frac{w-i}{w-i\mu}. \qquad (12.16)$$

Expansion (3.14) can be written in the explicit form

$$\frac{w-i}{w-i\mu}G_1(w)=H_1(w)+H_2(w); \qquad (12.17)$$

here

$$H_1(w)=\frac{(w-i)\,G_1(w)-(i\mu-i)\,G_1(i\mu)}{w-i\mu}, \qquad (12.18)$$

$$H_2(w)=\frac{(i\mu-i)\,G_1(i\mu)}{w-i\mu}. \qquad (12.19)$$

Function F(w) will be equal to

$$F(w)=\frac{(w^2+1)\,G_1(w)-i\,(\mu-1)\,G_1(i\mu)\,(w+i)}{w^2+\mu^2}, \qquad (12.20)$$

and f(x) will be expressed through F(w) according to general formula (3.18).

The same result we would obtain, also, using the second (algebraic) method, presented in paragraph 6. In formula (6.40) we must assume

$$F^*=F,\ G_1^*=H_1=G_1,\ \psi_1=1,\ w_1=i\mu,\ n=1,\ b=1. \quad (12.21)$$

Polynomial P_0, which is reduced to the constant, directly it is determined from (6.46):

$$P_0=-(i\mu-i)\,G_1(i\mu). \qquad (12.22)$$

After the substitution of values (12.21) and (12.22) into general formula (6.40) or (6.41) it passes in (12.20).

Page 450.

Case of 2 (2λ>1). Let us assume

$$2\lambda = 1 + \nu^2 \qquad (\nu > 0). \tag{12.23}$$

In this case general formula (6.41) is reduced to the following:

$$F(w) = \frac{(w^2+1)\,G_1(w) + (w+i)\,P_0}{w^2 - \nu^2}. \tag{12.24}$$

the numerator of our formula must become zero both when $w = +\nu$, and when $w = -\nu$. This leads to the relationship/ratio

$$(\nu - i)\,G_1(\nu) + (\nu + i)\,G_1(-\nu) = 0, \tag{12.25}$$

which expresses by itself the condition of orthogonality g(x) to the function

$$\gamma_1(x) = \cos\nu x + \frac{\sin\nu x}{\nu}, \tag{12.26}$$

which satisfies uniform integral equalization.

Finally, if $2\lambda = 1$, $\nu = 0$, then function 1-K(w) has the double root of w=0 and the absolute term of integral equation it must be orthogonal to the function

$$\gamma_1(x) = 1 + x. \tag{12.27}$$

Some important tasks of mathematical physics reduce to the integral equations of the form examined with the more complex nucleus. Thus, for instance, task of the coastal refraction of electromagnetic waves is given, as we saw in Chapter 18, to the equation

$$f(x) = g(x) + \frac{\lambda}{\pi}\int_0^\infty K_0(\mu|x-y|)\,f(y)\,dy, \tag{12.28}$$

where K_0 - MacDonald's function

$$K_0(\mu|x|) = \int_\mu^\infty e^{-|x|t}\frac{dt}{\sqrt{t^2-\mu^2}}. \tag{12.29}$$

In this case

$$K(w)=\frac{\lambda}{\sqrt{w^2+\mu^2}}. \qquad (12.30)$$

Let us give without the conclusion/output the solution of this equation for the case

$$g(x)=e^{-\mu x\cos\alpha}, \qquad (12.31)$$

where α - certain parameter.

In the physical task μ is a complex number with the positive real part, and λ - certain complex parameter. For simplicity let us assume μ=1 and let us introduce the designations

$$w=i\cos\tau,\quad \lambda=\sin\sigma. \qquad (12.32)$$

Page 451.

Then

$$\psi_1(w)=\sqrt{\frac{\cos\tau+1}{\cos\tau+\cos\sigma}}\exp\left(\frac{1}{2\pi}\int\limits_{\tau-\sigma}^{\tau+\sigma}\frac{u}{\sin u}\,du\right). \qquad (12.33)$$

This function satisfies the relationship/ratio

$$\psi_1(i\cos\tau)\,\psi_1[i\cos(\pi-\tau)]=\frac{\sin\tau}{\sin\tau-\sin\sigma}, \qquad (12.34)$$

which corresponds to formula (3.09). According to (12.05)

$$f(x)=\frac{1}{2\pi i}\int\limits_{\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty}e^{x\cos\tau}\,\frac{\psi_1(i\cos\tau)\,\psi_1(i\cos\alpha)}{\cos\tau+\cos\alpha}\sin\tau\,d\tau. \qquad (12.35)$$

Is hence easy to obtain different approximation formulas, valid for large x or for small ones σ. In this case it is necessary to have in mind that in the physical task equation itself (12.28) will be approximate, so that its strict solution has mainly mathematical interest; it have are given we as the illustration of our method.

Another example from mathematical physics represents the task of absorption and scattering of light in the atmosphere. This task was the object/subject of numerous experiments (Miln [36], Hopf [37, 38], Ambartsumnian [39]). The integral equation of task was for the first time comprised by Khvol'son. It has a kernel

$$k(x)=\frac{\lambda}{2}\int\limits_{|x|}^{\infty}\frac{e^{-t}}{t}\,dt. \qquad (12.36)$$

The converted kernel is equal

$$K(w)=\lambda\frac{\operatorname{arc\,tg} w}{w}. \qquad (12.37)$$

In this task λ it is real, it is positive and not more than unity (case λ=1 corresponds to pure/clean absorption). Function g(x) is equal to

$$g(x)=e^{-tx}\ (t\geqslant 1) \qquad (12.38)$$

or represented in the form

$$g(x)=\int\limits_{1}^{\infty}e^{-tx}\varphi(t)\,dt. \qquad (12.39)$$

Page 452.

Function K(w)of -1 has in the band where it is holomorphic, purely imaginary roots w=+-iμ, where

$$\frac{1}{\lambda}=\frac{1}{2\mu}\lg\frac{1+\mu}{1-\mu}\quad(0<\mu<1). \qquad (12.40)$$

General theory is directly added to this equation. In particular, it makes it possible to find approximation for f(x) with large x. In the case (12.38) we have

$$f(x)=C(t)\,e^{-\mu x}+f_2(x,\ t), \qquad (12.41)$$

where μ is determined from (12.40), and function $f_2(x, t)$ is such, that the product

$$e^{(1-\varepsilon)x}f_2(x,\ t)$$

remains limited for as small as desired ε. The constant C(t) is equal to

$$C(t)=-\lambda\frac{d\mu}{d\lambda}\frac{1}{t-\mu}\frac{\psi_1(it)}{\psi_1(i\mu)}. \qquad (12.42)$$

Function $\psi_1(it)$ for real t is real, and its logarithm is equal to

$$\lg\psi_1(it)=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\lg\left(1-\lambda\frac{\operatorname{arctg}u}{u}\right)\frac{t\,du}{u^2+t^2}. \qquad (12.43)$$

In the case (12.39) we will have

$$f(x)=Ce^{-\mu x}+f_2(x), \qquad (12.44)$$

where f_2 decreases more rapidly than the written out term.

The constant C is equal to

$$C=\int_{1}^{\infty} C(t)\,\varphi(t)\,dt. \qquad (12.45)$$

The intensity of light, which emerges from the atmosphere in this direction is proportional to value

$$I=\int_{0}^{\infty} e^{-xt'}f(x)\,dx=F(it'), \qquad (12.46)$$

where t' - secant of the angle between the data by direction and by normal to the atmosphere [parameter t in (12.38) there is a secant of the angle between the same standard/normal and direction of incident light]. Therefore function F(it')which according to (12.04) is equal to

$$F(it')=\frac{\psi_1(it)\,\psi_1(it')}{t+t'}, \qquad (12.47)$$

has direct physical value.

Page 453.

13. Summary of results.

The basic result of our research can be summarized as follows.

Let be is given the integral equation

$$f(x)=g(x)+\int_{0}^{\infty} k(|x-y|)\,f(y)\,dy \qquad (13.01)$$

with the symmetrical kernel, which depends on the absolute value of a difference in two arguments.

Relative to kernel k(x) we let us assume that not only it auto, but also the function

$$k_1(x) = e^{c|x|}k(x) \qquad (13.02)$$

will be for certain c>0 of that absolutely integrated also with the bounded variation in the infinite gap/interval. We introduce into the examination the function

$$K(w) = \int_{-\infty}^{+\infty} e^{iwx}k(x)\,dx. \qquad (13.03)$$

It will be even function from w, holomorphic within the band

$$-c < \operatorname{Im}(w) < +c, \qquad (13.04)$$

limited and continuous on the boundaries of this band. At infinity within and on the band edges function K(w) will decrease at least is inversely proportionally $|w|$.

Let us assume that the equation

$$K(w) - 1 = 0 \qquad (13.05)$$

does not have real roots. Then, if g(x) function absolutely integrated also with the bounded variation in the infinite gap/interval, then there is a solution f(x) of the integral equation

proposed, which possesses the following properties: it presents the sum of function with the bounded variation and to continuous function, remains everywhere limited and approaches at infinity zero. This is - unique solution with such properties. It can be represented in the form of the integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixw} F(w)\, dw, \qquad (13.06)$$

where F(w) is determined by formulas (3.16), (3.14), (3.09), (1.11).

Page 454.

Let us assume now that equations (13.05) it has 2n the real roots of multiplicity not higher than s. Then, if function g(x) is absolutely integrated and has the bounded variation in the infinite gap/interval, and product $(\lg x)^{\cdot} x^{s-1} g(x)$ it is absolutely integrated and if g(x) satisfies n to the conditions of the orthogonality of the form

$$\int_{0}^{+\infty} g(x)\gamma_r(x)\, dx = 0 \qquad (r = 0, 1, \ldots, n-1), \qquad (13.07)$$

Then, as in the preceding case, equation (13.01) has the unique solution which remains limited and approaches at infinity zero.

We pass to the homogeneous equation

$$f(x) = \int_{0}^{\infty} k(|x-y|) f(y)\, dy. \qquad (13.08)$$

If equation (13.05) does not have real roots and if v_0 - alleged part of the complex root nearest to the real axis or the singular point of function K(w) -1, then homogeneous equation (13.08) does not have solutions which would satisfy the inequality

$$|f(x)| < e^{vx}L \qquad (v < v_0). \qquad (13.09)$$

But if equation (13.05) has 2n real roots, then there exists exactly n of the linearly independent solutions

$$f(x) = \gamma_r(x) \qquad (r = 0, 1, \ldots, n-1) \qquad (13.10)$$

of homogeneous equation, which satisfy condition (13.09). These solutions enter into the conditions of orthogonality. They are given by formulas (9.02).

The results of present chapter for the first time have published we in the short presentation in work [40].

Page 455.

Addition 2.

THEORY TABLES OF AIRY'S FUNCTIONS.

In the first part of addition 2 are given representations of Airy's functions in the form of series/rows and integrals, and also asymptotic expressions for them. Are investigated the properties of Airy's functions in the complex plane and is established/installed their connection/communication with the Bessel functions; they are investigated also their roots. Is studied the use/application of Airy's functions to the asymptotic integration of the linear differential second order equations. For the Hankel functions, order and argument of which are great and close to each other, are derived/concluded the asymptotic representations through the Airy's functions.

The second part of addition 2 contains detailed (through 0.02) four-place tables of both Airy's functions u(t) and v(t), and also from the derivatives u'(t) and v'(t) in interval (-9<t<+9), i.e., beginning from those values where they have an oscillatory nature, and ending with those, where they have exponential character. Tables

are so detailed that in the majority of the cases they allow/assume linear interpolation.

Introduction.

In our works on the theory of diffraction and propagation of the electromagnetic waves, assembled in this book, extensively are used the Airy's functions. We considered it therefore appropriate to include/connect in this edition of the table of Airy's functions, after presupposing to them the short survey/coverage of the properties of these functions and their possible applications/appendices.

1. Determination and the basic properties of Airy's functions.

Under the Airy's functions we will understand the functions, connected with Airy's known integral

$$v(t)=\frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{x^3}{3}+xt\right)dx, \qquad (1.01)$$

which is for the first time examined into 1838 in the research by Airy "About the intensity of light in the vicinities of caustic surface" [41].

Page 456.

Airy's integral is one of the solutions of the differential equation

$$w''(t) = tw(t) \qquad (1.02)$$

(namely that which decreases at positive infinity more rapidly than any final degree of t). Together with this solution of v(t) we will examine another, linearly independent, solution u(t)which will be more precise determined below. Functions u(t) and v(t) we will call Airy's functions.

Let us consider the integral

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{tz - \frac{1}{3}z^3}\, dz, \qquad (1.03)$$

in which the way of integration Γ goes in the plane by complex variable z on ray/beam $\arc z = -\frac{2\pi}{3}$ from infinity to zero and on ray/beam arc z=0 (along the real axis) from zero to.

Integral (1.03) descends at all complex values of t and represents whole transcendental function from t. It is easy to check that the determined by it function w(t) satisfies differential equation (1.02). With t=0 function w(t) and its derivative w'(t) accept the values

$$w(0) = \frac{2\sqrt{\pi}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} e^{i\frac{\pi}{6}} = 1{,}0899290710 + i0{,}6292708425, \quad (1.04)$$

$$w'(0) = \frac{2\sqrt{\pi}}{3^{4/3}\Gamma\left(\frac{4}{3}\right)} e^{-i\frac{\pi}{6}} = 0{,}7945704238 - i0{,}4587454481. \quad (1.05)$$

Function w(t) as whole transcendental function is decomposed/expanded in the power series, which converges at all values of t. This series/row takes the form

$$w(t) = w(0)\left\{1 + \frac{t^3}{2\cdot 3} + \frac{t^6}{(2\cdot 5)\cdot(3\cdot 6)} + \frac{t^9}{(2\cdot 5\cdot 8)\cdot(3\cdot 6\cdot 9)} + \cdots\right\} +$$
$$+ w'(0)\left\{t + \frac{t^4}{3\cdot 4} + \frac{t^7}{(3\cdot 6)\cdot(4\cdot 7)} + \frac{t^{10}}{(3\cdot 6\cdot 9)\cdot(4\cdot 7\cdot 10)} + \cdots\right\}. \quad (1.06)$$

Counting t real, let us separate/liberate in w(t) real and alleged part, after assuming

$$w(t) = u(t) + iv(t). \qquad (1.07)$$

Functions u(t) and V(t) will be two independent integrals of equation (1.02), connected with the relationship/ratio

$$u'(t)\, v(t) - u(t)\, v'(t) = 1. \qquad (1.08)$$

Page 457.

In this case function v(t), the defined as alleged part w(t), will coincide with Airy's integral (1.01). We are right to call therefore functions u(t) and v(t) by Airy's functions.

Both Airy's functions, real at the real values of argument t, are the whole transcendental functions, which are determined for the

complex values of t. In this case occur the relationships/ratios:

$$w(t) = u(t) + iv(t), \quad (1.09)$$

$$w\left(te^{i\frac{\pi}{3}}\right) = 2e^{i\frac{\pi}{6}}v(-t), \quad (1.10)$$

$$w\left(te^{i\frac{2\pi}{3}}\right) = e^{i\frac{\pi}{3}}[u(t) - iv(t)], \quad (1.11)$$

$$w(te^{i\pi}) = u(-t) + iv(-t), \quad (1.12)$$

$$w\left(te^{i\frac{4}{3}\pi}\right) = 2e^{i\frac{\pi}{6}}v(t), \quad (1.13)$$

$$w\left(te^{i\frac{5\pi}{3}}\right) = e^{i\frac{\pi}{3}}[u(-t) - iv(-t)]. \quad (1.14)$$

These relationships/ratios express the values of function w(t) on six rays/beams $\arg t = \frac{n\pi}{3}$ $(n = 0, 1, 2, 3, 4, 5)$ through the real Airy's functions u(t) and v(t).

2. Asymptotic expressions for the Airy's functions.

We will count t by large positive number and let us assume

$$x = \frac{2}{3}t^{3/2}. \quad (2.01)$$

Let us designate by symbol F_{20} (α, β, z) the formal series/row, comprised according to the law

$$F_{20}(\alpha, \beta, z) = 1 + \frac{\alpha\beta}{1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2}z^2 + \cdots \quad (2.02)$$

Then for the Airy's functions occur the following asymptotic expressions:

$$u(t) = t^{-\frac{1}{4}}e^{x}F_{20}\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2x}\right), \quad (2.03)$$

$$u'(t) = t^{\frac{1}{4}}e^{x}F_{20}\left(-\frac{1}{6}, \frac{7}{6}, \frac{1}{2x}\right), \quad (2.04)$$

$$v(t) = \frac{1}{2}t^{-\frac{1}{4}}e^{-x}F_{20}\left(\frac{1}{6}, \frac{5}{6}, -\frac{1}{2x}\right), \quad (2.05)$$

$$v'(t) = -\frac{1}{2}t^{\frac{1}{4}}e^{-x}F_{20}\left(-\frac{1}{6}, \frac{7}{6}, -\frac{1}{2x}\right). \quad (2.06)$$

Page 458.

For the negative values of the argument of asymptotic ones the expressions of Airy's functions will be obtained by the isolation/evolution of real and alleged part in the formulas

$$w(-t) = t^{-\frac{1}{4}} e^{i\left(x+\frac{\pi}{4}\right)} F_{20}\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2ix}\right), \qquad (2.07)$$

$$w'(-t) = t^{\frac{1}{4}} e^{i\left(x-\frac{\pi}{4}\right)} F_{20}\left(-\frac{1}{6}, \frac{7}{6}, \frac{1}{2ix}\right). \qquad (2.08)$$

The expressions given here are valid not only at the real positive values of t, but also in certain sector, which switches on positive real axis. This sector is various for different functions, but in any case all resulting expressions are valid under the condition

$$-\frac{\pi}{3} < \operatorname{arc} t < \frac{\pi}{3}.$$

If we assume

$$F_{20}\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2x}\right) = 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots, \qquad (2.09)$$

that coefficients of a_1, a_2, ... will be equal to

$$\left.\begin{aligned} a_1 = \frac{5}{72};\ a_2 = \frac{(5\cdot 11)\cdot 7}{1\cdot 2\cdot (72)^2};\ a_3 = \frac{(5\cdot 11\cdot 17)\,(7\cdot 13)}{1\cdot 2\cdot 3\cdot (72)^3};\\ a_n = \frac{5\cdot 11\ \ldots\ (6n-1)\cdot 7\cdot 13\ \ldots\ (6n-5)}{1\cdot 2\ \ldots\ n\cdot (72)^n}. \end{aligned}\right\} \qquad (2.10)$$

It is analogous in the series/row

$$F_{20}\left(-\frac{1}{6},\ \frac{7}{6},\ \frac{1}{2x}\right)=1-\frac{b_1}{x}-\frac{b_2}{x^2}-\frac{b_3}{x^3}-\cdots \quad (2.11)$$

coefficients b_1, b_2, b_3 ... are equal to

$$\left.\begin{aligned} b_1=\frac{7}{72};\ b_2=\frac{(7\cdot 13)\cdot 5}{1\cdot 2\ (72)^2};\ b_3=\frac{(7\cdot 13\cdot 19)\cdot(5\cdot 11)}{1\cdot 2\cdot 3\cdot(72)^3};\\ b_n=\frac{7\cdot 13\ \ldots\ (6n+1)\cdot 5\cdot 11\ \ldots\ (6n-7)}{1\cdot 2\ \ldots\ n\cdot(72)^n}. \end{aligned}\right\} \quad (2.12)$$

In the explicit form asymptotic expressions for the Airy's functions from the positive argument will be written:

$$u(t)=t^{-\frac{1}{4}}e^{x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right), \quad (2.13)$$

$$u'(t)=t^{\frac{1}{4}}e^{x}\left(1-\frac{b_1}{x}-\frac{b_2}{x^2}-\cdots\right), \quad (2.14)$$

$$v(t)=\frac{1}{2}t^{-\frac{1}{4}}e^{-x}\left(1-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\cdots\right), \quad (2.15)$$

$$v'(t)=-\frac{1}{2}t^{\frac{1}{4}}e^{-x}\left(1+\frac{b_1}{x}-\frac{b_2}{x^2}+\frac{b_3}{x^3}-\cdots\right). \quad (2.16)$$

Page 459.

Corresponding expressions for the Airy's functions from the negative argument will be:

$$u(-t) = t^{-\frac{1}{4}} \cos\left(x + \frac{\pi}{4}\right) \left[1 - \frac{a_2}{x^2} + \frac{a_4}{x^4} - \frac{a_6}{x^6} + \cdots\right] +$$

$$+ t^{-\frac{1}{4}} \sin\left(x + \frac{\pi}{4}\right) \left[\frac{a_1}{x} - \frac{a_3}{x^3} + \frac{a_5}{x^5} - \frac{a_7}{x^7} + \cdots\right], \quad (2.17)$$

$$u'(-t) = t^{\frac{1}{4}} \sin\left(x + \frac{\pi}{4}\right) \left[1 + \frac{b_2}{x^2} - \frac{b_4}{x^4} + \frac{b_6}{x^6} - \cdots\right] +$$

$$+ t^{\frac{1}{4}} \cos\left(x + \frac{\pi}{4}\right) \left[\frac{b_1}{x} - \frac{b_3}{x^3} + \frac{b_5}{x^5} - \frac{b_7}{x^7} + \cdots\right], \quad (2.18)$$

$$v(-t) = t^{-\frac{1}{4}} \sin\left(x + \frac{\pi}{4}\right) \left[1 - \frac{a_2}{x^2} + \frac{a_4}{x^4} - \frac{a_6}{x^6} + \cdots\right] -$$

$$- t^{-\frac{1}{4}} \cos\left(x + \frac{\pi}{4}\right) \left[\frac{a_1}{x} - \frac{a_3}{x^3} + \frac{a_5}{x^5} - \frac{a_7}{x^7} + \cdots\right], \quad (2.19)$$

$$v'(-t) = -t^{\frac{1}{4}} \cos\left(x + \frac{\pi}{4}\right) \left[1 + \frac{b_2}{x^2} - \frac{b_4}{x^4} + \frac{b_6}{x^6} + \cdots\right] +$$

$$+ t^{\frac{1}{4}} \sin\left(x + \frac{\pi}{4}\right) \left[\frac{b_1}{x} - \frac{b_3}{x^3} + \frac{b_5}{x^5} - \frac{b_7}{x^7} + \cdots\right]. \quad (2.20)$$

3. Connection/communication of Airy's functions with the Bessel functions.

Airy's functions from the positive argument are expressed as the Bessel functions of first and second order on the order of 1/3 from the alleged argument. Airy's functions from the negative argument are expressed as the Bessel functions of first and second order on the order of 1/3 from the real argument. Finally, complex Airy's function W simply is expressed as the first Hankel function on the order of 1/3. Derivatives of Airy's functions are expressed as the corresponding Bessel functions and Hankel on the order of 2/3.

Page 460.

We accept for the functions of Bessel and Hankel, Watson's designations [33]. Counting t>0 and assuming/setting $x=\frac{2}{3}t^{3/2}$, we will then have:

$$u(t)=\sqrt{\frac{\pi}{3}t}\,[I_{-1/3}(x)+I_{1/3}(x)]=$$
$$=\sqrt{\frac{\pi}{3}t}\left[2I_{1/3}(x)+\frac{\sqrt{3}}{\pi}K_{1/3}(x)\right], \tag{3.01}$$
$$u(-t)=\sqrt{\frac{\pi}{3}t}\,[J_{-1/3}(x)-J_{1/3}(x)]=$$
$$=-\sqrt{\frac{\pi}{3}t}\left[\frac{1}{2}J_{1/3}(x)+\frac{\sqrt{3}}{2}Y_{2/3}(x)\right], \tag{3.02}$$
$$u'(t)=\sqrt{\frac{\pi}{3}}t\,[I_{-2/3}(x)+I_{2/3}(x)]=$$
$$=\sqrt{\frac{\pi}{3}}t\left[2I_{2/3}(x)+\frac{\sqrt{3}}{\pi}K_{2/3}(x)\right], \tag{3.03}$$
$$u'(-t)=\sqrt{\frac{\pi}{3}}t\,[J_{-2/3}(x)+J_{2/3}(x)]=$$
$$=\sqrt{\frac{\pi}{3}}t\left[\frac{1}{2}J_{2/3}(x)-\frac{\sqrt{3}}{2}Y_{2/3}(x)\right], \tag{3.04}$$
$$v(t)=\frac{1}{3}\sqrt{\pi t}\,[I_{-1/3}(x)-I_{1/3}(x)]=\frac{1}{\sqrt{3}\pi}\sqrt{t}\,K_{1/3}(x), \tag{3.05}$$
$$v(-t)=\frac{1}{3}\sqrt{\pi t}\,[J_{-1/3}(x)+J_{1/3}(x)]=$$
$$=\sqrt{\frac{\pi}{3}t}\left[\frac{\sqrt{3}}{2}J_{1/3}(x)-\frac{1}{2}Y_{1/3}(x)\right], \tag{3.06}$$
$$v'(t)=-\frac{1}{3}\sqrt{\pi}\,t\,[I_{-2/3}(x)-I_{2/3}(x)]=-\frac{1}{\sqrt{3}\pi}tK_{2/3}(x), \tag{3.07}$$
$$v'(-t)=-\frac{1}{3}\sqrt{\pi}\,t\,[J_{-2/3}(x)-J_{2/3}(x)]=$$
$$=\sqrt{\frac{\pi}{3}}t\left[\frac{\sqrt{3}}{2}J_{2/3}(x)+\frac{1}{2}Y_{2/3}(x)\right], \tag{3.08}$$
$$w(-t)=\sqrt{\frac{\pi}{3}}e^{i\frac{2\pi}{3}}\sqrt{t}\,H^{(1)}_{1/3}(x), \tag{3.09}$$
$$w'(-t)=\sqrt{\frac{\pi}{3}}e^{i\frac{\pi}{3}}tH^{(1)}_{2/3}(x). \tag{3.10}$$

Page 461.

It is important to note that the Airy's functions, as the functions in question from t, essence whole transcendental functions, whereas the right sides of the previous formulas (and also the

entering in them Bessel functions and Hankel) will not be integral functions of x, but will have with x=0 singular point. This difference is manifested at the low values of argument in more smooth running of tables for the Airy's functions in comparison with the tables for the functions of Bessel, which considerably facilitates interpolation. The whole transcendental character of Airy's functions considerably simplifies also reasonings in the theoretical studies.

4. Roots of Airy's functions.

In the applications/appendices are most frequently encountered the roots of function v(t) and by its derivative v'(t). Since with negative t of Airy's function they have an oscillatory nature, then these roots are real and negative. Let us designate roots of v(t) through $-\tau_s^0$ and roots of v'(t) - through $-\tau_s'$, where τ_s^0 and τ_s' – positive values. the values of the first five roots and their common logarithms are given in the following table:

s	τ_s^0	$\log \tau_s^0$	τ_s'	$\log \tau_s'$
1	2,33811	0,368864	1,01879	0,008086
2	4,08795	0,611506	3,24820	0,511642
3	5,52056	0,741983	4,82010	0,683056
4	6,78671	0,831659	6,16331	0,789814
5	7,94417	0,900048	7,37218	0,867506

Further roots can be calculated by using the formulas for the roots of the Bessel functions and Neumann and their linear

combinations (see Watson's book [33]). We have

$$\tau_s^0 = \left(\frac{3}{2}x_s^0\right)^{\frac{2}{3}}; \quad \tau_s' = \left(\frac{3}{2}x_s'\right)^{\frac{2}{3}}, \tag{4.01}$$

where x_s^0 and x_s' satisfy the equations

$$\frac{\sqrt{3}}{2}J_{1/3}(x_s^0) - \frac{1}{2}Y_{1/3}(x_s^0) = 0, \tag{4.02}$$

$$\frac{\sqrt{3}}{2}J_{2/3}(x_s') + \frac{1}{2}Y_{2/3}(x_s') = 0. \tag{4.03}$$

Page 462.

For values x_s^0 and x_s' are known the following approximations:

$$x_s^0 = \left(s - \frac{1}{4}\right)\pi + \frac{0{,}0884194}{4s-1} - \frac{0{,}08328}{(4s-1)^3} + \frac{0{,}4065}{(4s-1)^5}; \tag{4.04}$$

$$x_s' = \left(s - \frac{3}{4}\right)\pi - \frac{0{,}1237872}{4s-3} + \frac{0{,}07758}{(4s-3)^3} - \frac{0{,}389}{(4s-3)^5}. \tag{4.05}$$

These formulas are very precise even for the small values of s. Using relationships/ratios (4.01), we easily obtain hence values τ_s^0 and τ_s'.

Similar formulas can be used for the determination of roots $\tau = \tau_s^0(\sigma)$ of the equation

$$v(-\tau)\cos\pi\sigma - u(-\tau)\sin\pi\sigma = 0, \tag{4.06}$$

and also the roots $\tau = \tau_s'(\sigma)$ of the equation

$$v'(-\tau)\cos\pi\sigma - u'(-\tau)\sin\pi\sigma = 0. \tag{4.07}$$

At not too low values of s values $\tau_s^0(\sigma)$ and $\tau_s'(\sigma)$ will be obtained from the previous formulas by replacement of s on s+σ, so that it is possible conditionally to write

$$\tau_s^0(\sigma) = \tau_{s+\sigma}^0; \quad \tau_s'(\sigma) = \tau_{s+\sigma}'. \tag{4.08}$$

In particular, with σ=1/2 we will have roots of u($-\tau$) and u'($-\tau$).

Roots t_s^0 of the complex function w(t), and also roots t_s' of its derivative w'(t) lie/rest on ray/beam arc t=π/3 and are expressed as roots τ_s^0 and τ_s' functions v(-t) and v'(-t) of the formulas

$$t_s^0 = \tau_s^0 e^{i\frac{\pi}{3}};\quad t_s' = \tau_s' e^{i\frac{\pi}{3}}. \tag{4.09}$$

Thus, the table given above simultaneously gives the moduli/modules of values t_s^0 and t_s'.

Page 463.

5. Use/application of Airy's functions to the asymptotic integration of the linear differential second order equation.

Airy's functions have numerous applications/appendices in mathematical physics, mainly in the theory of diffraction. The mathematical basis of the majority of these applications/appendices is the approximative integration of the equation

$$\frac{d^2y}{dx^2} = k^2 p(x)\, y, \tag{5.01}$$

in which k is the high parameter. If in certain gap/interval of change X function p(x) does not reverse the sign (or it satisfies

some general conditions), then the integral of equation (5.01) is approximated through the exponential or trigonometric functions.

With p>0 we have

$$y \cong \frac{C_1}{\sqrt[4]{p(x)}} \exp\left(k \int_{x_0}^{x} \sqrt{p(x)}\, dx\right) \qquad (5.02)$$

or

$$y \cong \frac{C_2}{\sqrt[4]{p(x)}} \exp\left(-k \int_{x_0}^{x} \sqrt{p(x)}\, dx\right), \qquad (5.03)$$

according to that, we do seek integral increasing or decreasing with increase $x-x_0$. To take the sum of expressions (5.02) and (5.03), generally speaking, does not have a sense, since if the constants C_1 and C_2 of one order, then the decreasing integral will be small not only in comparison with entire value of the increasing integral, but also in comparison with the error in this value, which occurs from the use of asymptotic expression.

With p<0 asymptotic expression for y takes the form

$$y = \frac{C_1'}{\sqrt[4]{-p(x)}} \cos\left(k \int_{x_0}^{x} \sqrt{-p(x)}\, dx\right) + \\ + \frac{C_2'}{\sqrt[4]{-p(x)}} \sin\left(k \int_{x_0}^{x} \sqrt{-p(x)}\, dx\right). \qquad (5.04)$$

Expressions (5.02)-(5.04) cease to be applicable, if within the gap/interval in question function p(x) becomes zero. If in this case

the root of function p(x) - is simple, then solution of equation (5.01) is approximated through the Airy's functions.

Let us replace in equation (5.01) variable/alternating, introducing the new independent variable/alternating ζ and new function z of the formulas

$$x = x(\zeta); \quad y(x) = \sqrt{\frac{dx}{d\zeta}}\, z(\zeta). \tag{5.05}$$

Page 464.

If y satisfied equation (5.01), then equation for z will be

$$\frac{d^2 z}{d\zeta^2} = \left[s(\zeta) + k^2 p(x) \left(\frac{dx}{d\zeta} \right)^2 \right] z, \tag{5.06}$$

where through $s(\zeta)$ is marked for the brevity the differential expression

$$s(\zeta) = -\frac{1}{2} \frac{d^2}{d\zeta^2} \left(\lg \frac{dx}{d\zeta} \right) + \frac{1}{4} \left[\frac{d}{d\zeta} \left(\lg \frac{dx}{d\zeta} \right) \right]^2, \tag{5.07}$$

which is conventionally designated as Schwarz's derivative.

Let $x=x_0$ be the simple root of function p(x), so that

$$p(x_0) = 0; \quad p'(x_0) \neq 0.$$

We will count for certainty $p'(x_0)>0$ and let us produce the substitution

$$k \int_{x_0}^{x} \sqrt{p(x)}\, dx = \frac{2}{3} t^{\frac{3}{2}} \quad (x > x_0,\ t > 0), \tag{5.08}$$

$$k \int_{x}^{x_0} \sqrt{-p(x)}\, dx = \frac{2}{3} (-t)^{\frac{3}{2}} \quad (x < x_0,\ t < 0). \tag{5.09}$$

If p(x) has continuous second derivative, then near $x=x_0$ both formulas give

$$t=k^{\frac{2}{3}}[p'(x_0)]^{\frac{1}{3}}(x-x_0)+\cdots, \qquad (5.10)$$

where the uncopied/unordered terms will be order $(x-x_0)^2$ and above. Hence, conversely, with an accuracy down to the terms of order t^2 it will be

$$x=x_0+k^{-\frac{2}{3}}[p'(x_0)]^{-\frac{1}{3}}t+\cdots. \qquad (5.11)$$

To values $x-x_0$ of the order of one correspond values t of order $k^{\frac{2}{3}}$.

from formulas (5.08)-(5.09) it escape/ensues

$$k^2p(x)\left(\frac{dx}{dt}\right)^2=t. \qquad (5.12)$$

Therefore, if we in equation (5.06) for z assume $\zeta=t$, it takes the form

$$\frac{d^2z}{dt^2}=(s+t)z. \qquad (5.13)$$

Page 465.

The estimation of function s(t) shows that it will be, generally speaking, order $k^{-\frac{4}{3}}$, to very small both with t, close to zero, and with the large ones of value $|t|$ (order $k^{\frac{2}{3}}$). Function s(t) can become

large only if we x will approach the following root [1] $x=x_1$ of function p(x).

FOOTNOTE [1]. The case of two roots of function p(x) is examined in the paragraph of 4 Chapters 15 of this book. ENDFOOTNOTE.

If we exclude this case, then in equation (5.13) it is possible to disregard in the coefficient with z value s, after which we will obtain

$$\frac{d^2z}{dt^2} = tz, \qquad (5.14)$$

i.e., the differential equation of Airy's functions.

Let us write the general solution of equation (5.14) in the form

$$z = Au(t) + Bv(t). \qquad (5.15)$$

Being returned to the initial function y(x), we will have

$$y = \frac{1}{\sqrt{k}} \sqrt[4]{\frac{t}{p(x)}}\,[Au(t) + Bv(t)], \qquad (5.16)$$

where t is determined from formulas (5.08) and (5.09).

Let us establish connection/communication between the obtained expression for y and previous expressions (5.02)-(5.04).

When difference $x-x_1$ is positive and final, value t will be positive and great. Examining such values of t, we must distinguish

two cases: $A \neq 0$ and A=0. When A is not equal to zero, we have the increasing integral whose asymptotic expression will be obtained, if we reject/throw in (5.16) the term, which contains v(t), and to replace u(t) by value

$$u(t) = t^{-\frac{1}{4}} \exp\left(\frac{2}{3} t^{\frac{3}{2}}\right), \qquad (5.17)$$

i.e., by the dominant term of the asymptotic expression, examined in paragraph 2.

Using formula (5.08), we will obtain

$$y = \frac{A}{\sqrt{k}\sqrt[4]{p(x)}} \exp\left(k \int_{x_0}^{x} \sqrt{p(x)}\, dx\right), \qquad (5.18)$$

i.e., expression form (5.02) with the value by the constant C_1, equal to $C_1 = A/\sqrt{k}$. The constant B into this expression does not enter; therefore different integrals can have one and the same asymptotic expression.

Page 466.

In the second case when constant A it is equal to zero, in expression (5.16) remains the second term. Substituting in it v(t) by value

$$v(t) = \frac{1}{2} t^{-\frac{1}{4}} \exp\left(-\frac{2}{3} t^{\frac{3}{2}}\right), \qquad (5.19)$$

we obtain

$$y=\frac{B}{2\sqrt{k}\sqrt[4]{p(x)}}\exp\left(-k\int_{x_0}^{x}\sqrt{p(x)}\,dx\right);\quad A=0. \qquad (5.20)$$

In this case we deal concerning the decreasing integral of form (5.03), moreover the constant C_2 is equal to $\frac{B}{2\sqrt{k}}$.

Passing to the finite negative values of $x-x_0$, to which correspond large negative t, we can not distinguish two cases ($A\neq0$ and A=0), since both Airy's functions will be one order. Using for these functions their approximations

$$u(t)=(-t)^{-\frac{1}{4}}\cos\left[\frac{2}{3}(-t)^{\frac{3}{2}}+\frac{\pi}{4}\right], \qquad (5.21)$$

$$v(t)=(-t)^{-\frac{1}{4}}\sin\left[\frac{2}{3}(-t)^{\frac{3}{2}}+\frac{\pi}{4}\right], \qquad (5.22)$$

we obtain as a result of (5.09) the following formula for y:

$$y=\frac{A}{\sqrt{k}\sqrt[4]{-p(x)}}\cos\left(k\int_{x}^{x_0}\sqrt{-p(x)}\,dx+\frac{\pi}{4}\right)+$$

$$+\frac{B}{\sqrt{k}\sqrt[4]{-p(x)}}\sin\left(k\int_{x}^{x_0}\sqrt{-p(x)}\,dx+\frac{\pi}{4}\right). \qquad (5.23)$$

This expression easily is reduced to the form (5.04), moreover the constants C'_1 and C'_2 are equal to

$$C'_1=\frac{A+B}{\sqrt{2k}};\quad C'_2=\frac{A-B}{\sqrt{2k}}. \qquad (5.24)$$

Thus, the expression obtained by us for the integral of equation (5.01) through the Airy's functions is led in the limiting cases to the simpler expressions through exponential and trigonometric

functions.

Essential advantage of the expression through the Airy's functions is the fact that it is applicably even in entire interval which switches on root of $x=x_0$ of function p(x), whereas the expressions through the elementary functions are applicable are only are sufficiently are far through the elementary functions applicable only sufficiently far from the root. As far as the suitability of our expression for the numerical calculations is concerned, after Airy's functions are tabulated, the use by them is not at all not more complicated than the use of tables for the elementary functions.

Page 467.

Many functions, which are encountered in mathematical physics, either themselves satisfy the equation of form (5.01), or they are led to similar. Therefore the approximations given here for them can have vast uses/applications. Thus, in quantum mechanics similar expressions were proposed by Kramers and were applied by it to the proof of formulas "half integral" quantizations. In theory of Bessel functions results presented here can serve for the derivation of the asymptotic formulas, suitable for that case when order and argument of Bessel function are great and close to each other. Similar expressions were given in our previous work [20].

6. Use/application of Airy's functions to the approximate representation of the Hankel functions.

As the second typical application/appendix of Airy's functions we will give the conclusion/output of asymptotic expression for the Hankel function $H_\nu^{(1)}(\rho)$, where order ν and argument ρ they are great and close to each other, in the sense that the relation

$$\frac{\nu-\rho}{\sqrt[3]{\frac{\rho}{2}}}=t \qquad (6.01)$$

remains limited.

In this conclusion/output is used not differential equation for the Airy's functions, but their representation in the form of definite integral (1.03).

The Hankel function $H_\nu^{(1)}(\rho)$ allows/assumes the integral representation

$$H_\nu^{(1)}(\rho)=\frac{1}{\pi i}\int\limits_C e^{-\rho\,\mathrm{sh}\,v+\nu v}\,dv, \qquad (6.02)$$

where duct/contour C goes on the straight line $Im(v)=-\pi$ from $-\pi i-\infty$ to certain point $v=v_0$ in third fourth of plane v (for example, $v_0=\frac{-\pi}{\sqrt{3}}-i\pi$), then on the straight line, which combines point $v=v_0$

since the origin of the coordinates v=0, and finally along the real axis from 0 to ∞. Is expressed according to (6.01) v' through t and let us introduce the variable/alternating of integration

M

$$z = v \sqrt[3]{\frac{\rho}{2}}. \tag{6.03}$$

Page 478.

Integrand in (6.02) can be represented in the form of the product of two factors: the factor

$$\exp\left[(\nu - \rho)\, v - \frac{\rho v^3}{6}\right] = \exp\left(tz - \frac{z^3}{3}\right), \tag{6.04}$$

which does not contain clearly ρ, and the factor

$$\exp\left[-\rho \operatorname{sh} v + \rho v + \rho \frac{v^3}{6}\right] = 1 - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} z^5 \ldots, \tag{6.05}$$

which, with final z and large ρ, can be decomposed according to fractional negative degrees ρ (by multiple -2/3).

Substituting these expressions into integral (6.02), we obtain

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i}\left(\frac{\rho}{2}\right)^{-\frac{1}{3}} \int_\Gamma e^{tz - \frac{1}{3} z^3}\left[1 - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} z^5 + \cdots\right] dz, \tag{6.06}$$

where Γ is a duct/contour in plane z, which corresponds to duct/contour C in plane v. In the main section this duct/contour Γ coincides with duct/contour Γ in formula (1.03). Computing integrals with the help of (1.03), we obtain

$$H_\nu^{(1)}(\rho) = -\frac{i}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-\frac{1}{3}}\left[w(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} w^{(5)}(t) + \cdots\right]. \tag{6.07}$$

By the force of differential equation (1.02) the fifth derivative is equal to

$$w^{(5)}(t) = t^2 w'(t) + 4tw(t). \tag{6.08}$$

Substituting this expression in (6.07), we obtain the unknown asymptotic expression for the Hankel function:

$$H_\nu^{(1)}(\rho) = -\frac{i}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-\frac{1}{3}}\left\{w(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} [t^2 w'(t) + 4tw(t)] + \cdots\right\}. \tag{6.09}$$

After dividing here real and alleged parts, we come to the asymptotic expressions for the Bessel functions and Neumann through the Airy's functions

$$J_\nu(\rho) = \frac{1}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-\frac{1}{3}}\left\{v(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} [t^2 v'(t) + 4tv(t)] + \cdots\right\}, \tag{6.10}$$

$$Y_\nu(\rho) = -\frac{1}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-\frac{1}{3}}\left\{u(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-\frac{2}{3}} [t^2 u'(t) + 4tu(t)] + \cdots\right\}. \tag{6.11}$$

Page 469.

The obtained expressions are valid and at the complex values of

t and can serve, in particular, for the approximate determination of the roots of Hankel function, considered as function from ν, which has a value in the theory of diffraction [1].

FOOTNOTE [1]. The tables of connected with v(t) function $Z_{1/3}(x)=\frac{\sqrt{3}}{2}J_{1/3}(x)-\frac{1}{2}Y_{1/3}(x)$ of argument x are given in Foch's article and Kolpino [42]. ENDFOOTNOTE.

7. Explanation of tables.

We give below tables of Airy's functions u(t), v(t) and their derivatives u'(t), v'(t). Initially tables were computed with a large number of signs, but results were rounded to four signs. For the negative values of t, where the Airy's functions have the oscillating character, are given four signs after comma. For the positive values of t, where the Airy's functions are monotonic, given four significant digits, if the first numeral is 2, 3, 4, 5, 6, 7, 8, 9, and five numerals, if the first numeral is 1. The values of functions are given for the values argument t from -9.00 to +9.00 through 0.02. This small tabular interval is accepted for facilitating interpolation. In the majority of the cases of sufficiently linear interpolation; in the exceptional cases it can be required interpolation with the second differences. Since together with the values of functions are given their first-order differences, then

interpolation on our tables is very simple.

For the positive values of the argument of function u(t) and u'(t) rapidly they grow, and functions v(t) and v'(t) rapidly are. Therefore in some intervals of the value of functions u(t) and u'(t), the divided into 10^3 and into 10^4, and values of functions v(t) and v'(t), multiplied by 10^3, 10^4 and 10^5.

For values of argument less than - 9.00 or large how +9.00, the Airy's functions and their derivatives are easily computed from the asymptotic expressions, given in paragraph 2; in these expressions it suffices to take two or three members.

Page 470

Tables of Airy functions.

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—9,00	0,5760	— 31	—0,1017	—1034	—0,0392	—345	—1,7293	101
—8,98	0,5729	— 51	—0,2051	—1023	—0,0737	—343	—1,7192	163
—8,96	0,5678	— 72	—0,3074	—1010	—0,1080	—338	—1,7029	224
—8,94	0,5606	— 92	—0,4084	— 994	—0,1418	—333	—1,6805	283
—8,92	0,5514	—111	—0,5078	— 973	—0,1751	—328	—1,6522	341
—8,90	0,5403	—131	—0,6051	— 949	—0,2079	—319	—1,6181	398
—8,88	0,5272	—149	—0,7000	— 922	—0,2398	—311	—1,5783	454
—8,86	0,5123	—167	—0,7922	— 893	—0,2709	—302	—1,5329	506
—8,84	0,4956	—185	—0,8815	— 859	—0,3011	—291	—1,4823	558
—8,82	0,4771	—202	—0,9674	— 823	—0,3302	—279	—1,4265	606
—8,80	0,4569	—218	—1,0497	— 784	—0,3581	—267	—1,3659	653
—8,78	0,4351	—233	—1,1281	— 743	—0,3848	—253	—1,3006	698
—8,76	0,4118	—247	—1,2024	— 699	—0,4101	—239	—1,2308	739
—8,74	0,3871	—262	—1,2723	— 653	—0,4340	—224	—1,1569	777
—8,72	0,3609	—273	—1,3376	— 606	—0,4564	—208	—1,0792	814
—7,70	0,3336	—285	—1,3982	— 555	—0,4772	—191	—0,9978	846
—8,68	0,3051	—296	—1,4537	— 503	—0,4963	—174	—0,9132	876
—8,66	0,2755	—306	—1,5040	— 451	—0,5137	—156	—0,8256	902
—8,64	0,2449	—314	—1,5491	— 395	—0,5293	—138	—0,7354	926
—8,62	0,2135	—321	—1,5886	— 340	—0,5431	—119	—0,6428	945
—8,60	0,1814	—327	—1,6226	— 284	—0,5550	—100	—0,5483	963
—8,58	0,1487	—333	—1,6510	— 226	—0,5650	— 81	—0,4520	975
—8,56	0,1154	—336	—1,6736	— 169	—0,5731	— 61	—0,3545	986
—8,54	0,0818	—340	—1,6905	— 111	—0,5792	— 41	—0,2559	991
—8,52	0,0478	—341	—1,7016	— 52	—0,5833	— 21	—0,1568	995
—8,50	0,0137	—341	—1,7068	+ 5	—0,5854	— 2	—0,0573	995

154

Page 471

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−8,50	+0,0137	−341	−1,7068	+ 5	−0,5854	− 2	−0,0573	995
−8,48	−0,0204	−341	−1,7063	64	−0,5856	+ 19	+0,0422	990
−8,46	−0,0545	−339	−1,6999	121	−0,5837	38	0,1412	984
−8,44	−0,0884	−335	−1,6878	177	−0,5799	57	0,2396	973
−8,42	−0,1219	−332	−1,6701	233	−0,5742	77	0,3369	960
−8,40	−0,1551	−327	−1,6468	288	−0,5665	96	0,4329	942
−8,38	−0,1878	−320	−1,6180	341	−0,5569	115	0,5271	923
−8,36	−0,2198	−313	−1,5839	393	−0,5454	133	0,6194	900
−8,34	−0,2511	−305	−1,5446	444	−0,5321	151	0,7094	874
−8,32	−0,2816	−295	−1,5002	493	−0,5170	168	0,7968	846
−8,30	−0,3111	−285	−1,4509	539	−0,5002	184	0,8814	814
−8,28	−0,3396	−273	−1,3970	585	−0,4818	200	0,9628	781
−8,26	−0,3669	−262	−1,3385	627	−0,4618	216	1,0409	744
−8,24	−0,3931	−248	−1,2758	667	−0,4402	230	1,1153	706
−8,22	−0,4179	−235	−1,2091	706	−0,4172	244	1,1859	665
−8,20	−0,4414	−220	−1,1385	741	−0,3928	257	1,2524	623
−8,18	−0,4634	−206	−1,0644	775	−0,3671	269	1,3147	578
−8,16	−0,4840	−189	−0,9869	804	−0,3402	280	1,3725	532
−8,14	−0,5029	−173	−0,9065	832	−0,3122	290	1,4257	484
−8,12	−0,5202	−156	−0,8233	857	−0,2832	299	1,4741	435
−8,10	−0,5358	−139	−0,7376	878	−0,2533	307	1,5176	385
−8,08	−0,5497	−121	−0,6498	897	−0,2226	315	1,5561	334
−8,06	−0,5618	−103	−0,5601	914	−0,1911	321	1,5895	282
−8,04	−0,5721	− 84	−0,4687	925	−0,1590	326	1,6177	229
−8,02	−0,5805	− 66	−0,3762	936	−0,1264	330	1,6406	176
−8,00	−0,5871	− 47	−0,2826	942	−0,0934	333	1,6582	123

155

Page 472

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−8,00	−0,5871	−47	−0,2826	942	−0,0934	333	1,6582	123
−7,98	−0,5918	−29	−0,1884	946	−0,0601	335	1,6705	69
−7,96	−0,5947	− 9	−0,0938	947	−0,0266	335	1,6774	16
−7,94	−0,5956	+10	+0,0009	944	+0,0069	336	1,6790	− 38
−7,92	−0,5946	28	0,0953	938	0,0405	334	1,6752	− 90
−7,90	−0,5918	47	0,1891	931	0,0739	332	1,6662	−143
−7,88	−0,5871	66	0,2822	919	0,1071	329	1 6519	−194
−7,86	−0,5805	84	0,3741	905	0,1400	324	1,6325	−246
−7,84	−0,5721	102	0,4646	888	0,1724	318	1,6079	−295
−7,82	−0,5619	119	0,5534	869	0,2042	313	1,5784	−343
−7,80	−0,5500	137	0,6403	846	0,2355	305	1,5441	−391
−7,78	−0,5363	153	0,7249	822	0,2660	296	1,5050	−436
−7,76	−0,5210	169	0,8071	795	0,2956	288	1,4614	−481
−7,74	−0,5041	185	0,8866	765	0,3244	277	1,4133	−523
−7,72	−0,4856	200	0,9631	733	0,3521	267	1,3610	−564
−7,70	−0,4656	215	1,0364	700	0,3788	255	1,3046	−602
−7,68	−0,4441	228	1,1064	664	0,4043	243	1,2444	−639
−7,66	−0,4213	240	1,1728	626	0,4286	229	1,1805	−673
−7,64	−0,3973	254	1,2354	587	0,4515	216	1,1132	−706
−7,62	−0,3719	264	1,2941	547	0,4731	201	1,0426	−736
−7,60	−0,3455	275	1,3488	503	0,4932	186	0,9690	−763
−7,58	−0,3180	284	1,3991	460	0,5118	171	0,8927	−788
−7,56	−0,2896	294	1,4451	416	0,5289	155	0,8139	−810
−7,54	−0,2602	301	1,4867	369	0,5444	138	0,7329	−831
−7,52	−0,2301	308	1,5236	323	0,5582	121	0,6498	−847
−7,50	−0,1993	314	1,5559	275	0,5703	105	0,5651	−863

156

Page 473

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—7,50	—0,1993	314	1,5559	275	0,5703	105	0,5651	—863
—7,48	—0,1679	319	1,5834	227	0,5808	87	0,4788	—874
—7,46	—0,1360	323	1,6061	179	0,5895	69	0,3914	—884
—7,44	—0,1037	326	1,6240	130	0,5964	52	0,3030	—890
—7,42	—0,0711	328	1,6370	81	0,6016	34	0,2140	—894
—7,40	—0,0383	330	1,6451	32	0,6050	16	0,1246	—896
—7,38	—0,0053	329	1,6483	— 17	0,6066	— 2	0,0350	—894
—7,36	0,0276	329	1,6466	— 64	0,6064	— 20	—0,0544	—890
—7,34	0,0605	327	1,6402	—113	0,6044	— 38	—0,1434	—884
—7,32	0,0932	324	1,6289	—160	0,6006	— 55	—0,2318	—874
—7,30	0,1256	321	1,6129	—207	0,5951	— 72	—0,3192	—863
—7,28	0,1577	316	1,5922	—252	0,5879	— 90	—0,4055	—848
—7,26	0,1893	310	1,5670	—297	0,5789	—106	—0,4903	—832
—7,24	0,2204	304	1,5373	—341	0,5683	—123	—0,5735	—813
—7,22	0,2507	297	1,5032	—383	0,5560	—139	—0,6548	—792
—7,20	0,2804	289	1,4649	—424	0,5421	—154	—0,7340	—769
—7,18	0,3093	280	1,4225	—463	0,5267	—170	—0,8109	—743
—7,16	0,3373	270	1,3762	—502	0,5097	—184	—0,8852	—716
—7,14	0,3643	260	1,3260	—538	0,4913	—199	—0,9568	—687
—7,12	0,3903	249	1,2722	—573	0,4714	—211	—1,0255	—655
—7,10	0,4152	237	1,2149	—606	0,4503	—225	—1,0910	—623
—7,08	0,4389	225	1,1543	—637	0,4278	—236	—1,1533	—588
—7,06	0,4614	211	1,0906	—665	0,4042	—248	—1,2121	—553
—7,04	0,4825	198	1,0241	—693	0,3794	—259	—1,2674	—515
—7,02	0,5023	184	0,9548	—717	0,3535	—269	—1,3189	—477
—7,00	0,5207	169	0,8831	—740	0,3266	—277	—1,3666	—437

157

Page 474

t	u	Δu		Δu'	v	Δv	v'	Δv'
−7,00	0,5207	169	0,8831	−740	0,3266	−277	−1,3666	−437
−6,98	0,5376	154	0,8091	−760	0,2989	−287	−1,4103	−397
−6,96	0,5530	139	0,7331	−779	0,2702	−293	−1,4500	−355
−6,94	0,5669	123	0,6552	−794	0,2409	−300	−1,4855	−313
−6,92	0,5792	107	0,5758	−808	0,2109	−307	−1,5168	−271
−6,90	0,5899	91	0,4950	−820	0,1802	−311	−1,5439	−227
−6,88	0,5990	75	0,4130	−828	0,1491	−315	−1,5666	−183
−6,86	0,6065	57	0,3302	−835	0,1176	−318	−1,5849	−139
−6,84	0,6122	41	0,2467	−839	0,0558	−321	−1,5988	− 96
−6,82	0,6163	24	0,1628	−841	0,0537	−322	−1,6084	− 51
−6,80	0,6187	8	0,0787	−841	0,0215	−323	−1,6135	− 7
−6,78	0,6195	− 10	−0,0054	−839	−0,0108	−323	−1,6142	+ 37
−6,76	0,6185	− 26	−0,0893	−833	−0,0431	−321	−1,6105	79
−6,74	0,6159	− 43	−0,1726	−826	−0,0752	−320	−1,6026	123
−6,72	0,6116	− 59	−0,2552	−817	−0,1072	−316	−1,5903	165
−6,70	0,6057	− 76	−0,3369	−806	−0,1388	−313	−1,5738	207
−6,68	0,5981	− 91	−0,4175	−792	−0,1701	−308	−1,5531	247
−6,66	0,5890	−107	−0,4967	−776	−0,2009	−303	−1,5284	288
−6,64	0,5783	−123	−0,5743	−759	−0,2312	−297	−1,4996	326
−6,62	0,5660	−137	−0,6502	−740	−0,2609	−289	−1,4670	364
−6,60	0,5523	−152	−0,7242	−718	−0,2898	−283	−1,4306	401
−6,58	0,5371	−166	−0,7960	−695	−0,3181	−273	−1,3905	436
−6,56	0,5205	−180	−0,8655	−670	−0,3454	−265	−1,3469	470
−6,54	0,5025	−193	−0,9325	−644	−0,3719	−255	−1,2999	502
−6,52	0,4832	−206	−0,9969	−616	−0,3974	−245	−1,2497	534
−6,50	0,4626	−217	−1,0585	−586	−0,4219	−233	−1,1963	563

158

Page 475

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
—6,50	0,4626	—217	—1,0585	—586	—0,4219	—233	—1,1963	563
—6,48	0,4409	—229	—1,1171	—556	—0,4452	—223	—1,1400	590
—6,46	0,4180	—240	—1,1727	—524	—0,4675	—210	—1,1810	617
—6,44	0,3940	—250	—1,2251	—490	—0,4885	—197	—1,0193	641
—6,42	0,3690	—260	—1,2741	—457	—0,5082	—185	—0,9552	664
—6,40	0,3430	—268	—1,3198	—421	—0,5267	—171	—0,8888	684
—6,38	0,3162	—276	—1,3619	—386	—0,5438	—157	—0,8204	703
—6,36	0,2886	—284	—1,4005	—348	—0,5595	—143	—0,7501	719
—6,34	0,2602	—290	—1,4353	—311	—0,5738	—128	—0,6782	735
—6,32	0,2312	—296	—1,4664	—274	—0,5866	—113	—0,6047	748
—6,30	0,2016	—301	—1,4938	—234	—0,5979	— 99	—0,5299	758
—6,28	0,1715	—306	—1,5172	—196	—0,6078	— 83	—0,4541	768
—6,26	0,1409	—309	—1,5368	—157	—0,6161	— 68	—0,3773	774
—6,24	0,1100	—311	—1,5525	—118	—0,6229	— 52	—0,2999	780
—6,22	0,0789	—314	—1,5643	— 78	—0,6281	— 36	—0,2219	782
—6,20	0,0475	—315	—1,5721	— 40	—0,6317	— 21	—0,1437	784
—6,18	0,0160	—315	—1,5761	0	—0,6338	— 6	—0,0653	782
—6,16	—0,0155	—315	—1,5761	39	—0,6344	+ 11	0,0129	780
—6,14	—0,0470	—314	—1,5722	76	—0,6333	26	0,0909	775
—6,12	—0,0784	—312	—1,5646	115	—0,6307	41	0,1684	768
—6,10	—0,1096	—309	—1,5531	153	—0,6266	57	0,2452	760
—6,08	—0,1405	—306	—1,5378	189	—0,6209	72	0,3212	750
—6,06	—0,1711	—301	—1,5189	225	—0,6137	86	0,3962	738
—6,04	—0,2012	—297	—1,4964	261	—0,6051	101	0,4700	723
—6,02	—0,2309	—291	—1,4703	295	—0,5950	116	0,5423	709
—6,00	—0,2600	—285	—1,4408	328	—0,5834	130	0,6132	691

159

Page 476

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
—6,00	—0,2600	—285	—1,4408	328	—0,5834	130	0,6132	691
—5,98	—0,2885	—278	—1,4080	362	—0,5704	143	0,6823	673
—5,96	—0,3163	—271	—1,3718	392	—0,5561	156	0,7496	652
—5,94	—0,3434	—262	—1,3326	423	—0,5405	170	0,8148	631
—5,92	—0,3696	—254	—1,2903	452	—0,5235	181	0,8779	608
—5,90	—0,3950	—244	—1,2451	480	—0,5054	194	0,9387	584
—5,88	—0,4194	—234	—1,1971	506	—0,4860	205	0,9971	559
—5,86	—0,4428	—224	—1,1465	531	—0,4655	216	1,0530	532
—5,84	—0,4652	—213	—1,0934	555	—0,4439	226	1,1062	505
—5,82	—0,4865	—202	—1,0379	577	—0,4213	236	1,1567	476
—5,80	—0,5067	—190	—0,9802	598	—0,3977	246	1,2043	446
—5,78	—0,5257	—178	—0,9204	617	—0,3731	254	1,2489	416
—5,76	—0,5435	—166	—0,8587	635	—0,3477	262	1,2905	385
—5,74	—0,5601	—152	—0,7952	651	—0,3215	269	1,3290	353
—5,72	—0,5753	—140	—0,7301	665	—0,2946	276	1,3643	321
—5,70	—0,5893	—126	—0,6636	678	—0,2670	283	1,3964	288
—5,68	—0,6019	—112	—0,5958	689	—0,2387	287	1,4252	254
—5,66	—0,6131	— 98	—0,5269	698	—0,2100	293	1,4506	221
—5,64	—0,6229	— 85	—0,4571	706	—0,1807	296	1,4727	187
—5,62	—0,6314	— 70	—0,3865	713	—0,1511	300	1,4914	152
—5,60	—0,6384	— 56	—0,3152	717	—0,1211	302	1,5066	119
—5,58	—0,6440	— 41	—0,2435	720	—0,0909	305	1,5185	84
—5,56	—0,6481	— 27	—0,1715	721	—0,0604	306	1,5269	50
—5,54	—0,6508	— 13	—0,0994	720	—0,0298	307	1,5319	16
—5,52	—0,6521	2	—0,0274	719	0,0009	306	1,5335	—17
—5,50	—0,6519	16	0,0445	715	0,0315	306	1,5318	—52

160

Page 477

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—5,50	—0,6519	16	0,0445	715	0,0315	306	1,5318	— 52
—5,48	—0,6503	30	0,1160	710	0,0621	305	1,5266	— 84
—5,46	—0,6473	44	0,1870	703	0,0926	302	1,5182	—118
—5,44	—0,6429	59	0,2573	695	0,1228	300	1,5064	—150
—5,42	—0,6370	72	0,3268	686	0,1528	296	1,4914	—181
—5,40	—0,6298	86	0,3954	674	0,1824	293	1,4733	—212
—5,38	—0,6212	99	0,4628	662	0,2117	288	1,4521	—243
—5,36	—0,6113	113	0,5290	648	0,2405	283	1,4278	—273
—5,34	—0,6000	125	0,5938	634	0,2688	277	1,4005	—301
—5,32	—0,5875	137	0,6572	616	0,2965	271	1,3704	—329
—5,30	—0,5738	150	0,7188	600	0,3236	264	1,3375	—357
—5,28	—0,5588	162	0,7788	580	0,3500	256	1,3018	—382
—5,26	—0,5426	173	0,8368	561	0,3756	249	1,2636	—408
—5,24	—0,5253	184	0,8929	540	0,4005	241	1,2228	—431
—5,22	—0,5069	194	0,9469	518	0,4246	231	1,1797	—455
—5,20	—0,4875	205	0,9987	496	0,4477	222	1,1342	—476
—5,18	—0,4670	214	1,0483	471	0,4699	212	1,0866	—497
—5,16	—0,4456	224	1,0954	448	0,4911	203	1,0369	—517
—5,14	—0,4232	232	1,1402	422	0,5114	191	0,9852	—534
—5,12	—0,4000	241	1,1824	397	0,5305	181	0,9318	—552
—5,10	—0,3759	248	1,2221	370	0,5486	170	0,8766	—567
—5,08	—0,3511	255	1,2591	343	0,5656	158	0,8199	—582
—5,06	—0,3256	262	1,2934	316	0,5814	147	0,7617	—594
—5,04	—0,2994	268	1,3250	287	0,5961	134	0,7023	—607
—5,02	—0,2726	273	1,3537	260	0,6095	122	0,6416	—617
—5,00	—0,2453	279	1,3797	231	0,6217	110	0,5799	—626

161

Page 478

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—5,00	—0,2453	279	1,3797	231	0,6217	110	0,5799	—626
—4,98	—0,2174	282	1,4028	202	0,6327	97	0,5173	—634
—4,96	—0,1892	287	1,4230	173	0,6424	84	0,4539	—640
—4,94	—0,1605	289	1,4403	144	0,6508	72	0,3899	—645
—4,92	—0,1316	292	1,4547	115	0,6580	58	0,3254	—649
—4,90	—0,1024	295	1,4662	86	0,6638	46	0,2605	—652
—4,88	—0,0729	295	1,4748	56	0,6684	33	0,1953	—653
—4,86	—0,0434	297	1,4804	28	0,6717	19	0,1300	—652
—4,84	—0,0137	296	1,4832	— 1	0,6736	7	0,0648	—651
—4,82	0,0159	297	1,4831	— 29	0,6743	— 7	—0,0003	—649
—4,80	0,0456	295	1,4802	— 58	0,6736	— 20	—0,0652	—644
—4,78	0,0751	294	1,4744	— 86	0,6716	— 32	—0,1296	—639
—4,76	0,1045	292	1,4658	—113	0,6684	— 45	—0,1935	—633
—4,74	0,1337	290	1,4545	—140	0,6639	— 58	—0,2568	—626
—4,72	0,1627	286	1,4405	—167	0,6581	— 70	—0,3194	—617
—4,70	0,1913	283	1,4238	—193	0,6511	— 82	—0,3811	—607
—4,68	0,2196	279	1,4045	—218	0,6429	— 94	—0,4418	—596
—4,66	0,2475	274	1,3827	—243	0,6335	—106	—0,5014	—584
—4,64	0,2749	269	1,3584	—267	0,6229	—118	—0,5598	—572
—4,62	0,3018	264	1,3317	—290	0,6111	—129	—0,6170	—557
—4,60	0,3282	257	1,3027	—314	0,5982	—140	—0,6727	—543
—4,58	0,3539	251	1,2713	—335	0,5842	—151	—0,7270	—527
—4,56	0,3790	244	1,2378	—356	0,5691	—161	—0,7797	—511
—4,54	0,4034	237	1,2022	—376	0,5530	—171	—0,8308	—493
—4,52	0,4271	229	1,1646	—395	0,5359	—181	—0,8801	—475
—4,50	0,4500	221	1,1251	—414	0,5178	—190	—0,9276	—457

162

Page 479

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
—4,50	0,4500	221	1,1251	—414	0,5178	—190	—0,9276	—457
—4,48	0,4721	212	1,0837	—432	0,4988	—199	—0,9733	—437
—4,46	0,4933	204	1,0405	—448	0,4789	—208	—1,0170	—417
—4,44	0,5137	194	0,9957	—464	0,4581	—215	—1,0587	—397
—4,42	0,5331	185	0,9493	—478	0,4366	—224	—1,0984	—375
—4,40	0,5516	176	0,9015	—493	0,4142	—230	—1,1359	—353
—4,38	0,5692	165	0,8522	—504	0,3912	—238	—1,1712	—332
—4,36	0,5857	155	0,8018	—517	0,3674	—244	—1,2044	—309
—4,34	0,6012	145	0,7501	—527	0,3430	—250	—1,2353	—286
—4,32	0,6157	134	0,6974	—536	0,3180	—255	—1,2639	—263
—4,30	0,6291	124	0,6438	—546	0,2925	—261	—1,2902	—240
—4,28	0,6415	112	0,5892	—552	0,2664	—265	—1,3142	—216
—4,26	0,6527	101	0,5340	—559	0,2399	—269	—1,3358	—193
—4,24	0,6628	90	0,4781	—565	0,2130	—273	—1,3551	—169
—4,22	0,6718	79	0,4216	—569	0,1857	—276	—1,3720	—144
—4,20	0,6797	67	0,3647	—573	0,1581	—278	—1,3864	—121
—4,18	0,6864	56	0,3074	—574	0,1303	—281	—1,3985	— 97
—4,16	0,6920	44	0,2500	—577	0,1022	—282	—1,4082	— 73
—4,14	0,6964	33	0,1923	—576	0,0740	—284	—1,4155	— 50
—4,12	0,6997	21	0,1347	—576	0,0456	—284	—1,4205	— 26
—4,10	0,7018	10	0,0771	—575	0,0172	—285	—1,4231	—2
—4,08	0,7028	— 2	0,0196	—572	—0,0113	—285	—1,4233	+ 21
—4,06	0,7026	—13	—0,0376	—569	—0,0398	—284	—1,4212	43
—4,04	0,7013	—25	—0,0945	—564	—0,0682	—282	—1,4169	67
—4,02	0,6988	—36	—0,1509	—559	—0,0964	—281	—1,4102	88
—4,00	0,6952	—47	—0,2068	—553	—0,1245	—280	—1,4014	111

163

Page 480

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
−4,00	0,6952	− 47	−0,2068	−553	−0,1245	−280	−1,4014	111
−3,98	0,6905	− 58	−0,2621	−546	−0,1525	−276	−1,3903	132
−3,96	0,6847	− 68	−0,3167	−538	−0,1801	−274	−1,3771	153
−3,94	0,6779	− 80	−0,3705	−530	−0,2075	−271	−1,3618	174
−3,92	0,6699	− 90	−0,4235	−520	−0,2346	−267	−1,3444	194
−3,90	0,6609	−100	−0,4755	−511	−0,2613	−263	−1,3250	213
−3,88	0,6509	−110	−0,5266	−499	−0,2876	−258	−1,3037	233
−3,86	0,6399	−120	−0,5765	−489	−0,3134	−254	−1,2804	251
−3,84	0,6270	−130	−0,6254	−476	−0,3388	−248	−1,2553	269
−3,82	0,6149	−140	−0,6730	−463	−0,3636	−243	−1,2284	287
−3,80	0,6009	−148	−0,7193	−450	−0,3879	−237	−1,1997	303
−3,78	0,5861	−157	−0,7643	−436	−0,4116	−231	−1,1694	319
−3,76	0,5704	−166	−0,8079	−422	−0,4347	−224	−1,1375	334
−3,74	0,5538	−174	−0,8501	−406	−0,4571	−217	−1,1041	349
−3,72	0,5364	−182	−0,8907	−392	−0,4788	−211	−1,0692	363
−3,70	0,5182	−190	−0,9299	−375	−0,4999	−202	−1,0329	377
−3,68	0,4992	−197	−0,9674	−359	−0,5201	−196	−0,9952	389
−3,66	0,4795	−204	−1,0033	−343	−0,5397	−187	−0,9563	401
−3,64	0,4591	−211	−1,0376	−326	−0,5584	−179	−0,9162	412
−3,62	0,4380	−217	−1,0702	−308	−0,5763	−171	−0,8750	422
−3,60	0,4163	−223	−1,1010	−291	−0,5934	−162	−0,8328	432
−3,58	0,3940	−229	−1,1301	−273	−0,6096	−154	−0,7896	441
−3,56	0,3711	−234	−1,1574	−255	−0,6250	−144	−0,7455	448
−3,54	0,3477	−239	−1,1829	−237	−0,6394	−136	−0,7007	457
−3,52	0,3238	−244	−1,2066	−219	−0,6530	−126	−0,6550	463
−3,50	0,2994	−247	−1,2285	−201	−0,6656	−117	−0,6087	468

164

Page 481

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—3,50	0,2994	—247	—1,2285	—201	—0,6656	—117	—0,6087	468
—3,48	0,2747	—252	—1,2486	—181	—0,6773	—108	—0,5619	474
—3,46	0,2495	—255	—1,2667	—164	—0,6881	— 98	—0,5145	478
—3,44	0,2240	—258	—1,2831	—145	—0,6979	— 89	—0,4667	482
—3,42	0,1982	—261	—1,2976	—126	—0,7068	— 78	—0,4185	485
—3,40	0,1721	—263	—1,3102	—108	—0,7146	— 70	—0,3700	487
—3,38	0,1458	—265	—1,3210	— 89	—0,7216	— 59	—0,3213	488
—3,36	0,1193	—267	—1,3299	— 71	—0,7275	— 50	—0,2725	489
—3,34	0,0926	—268	—1,3370	— 53	—0,7325	— 39	—0,2236	490
—3,32	0,0658	—269	—1,3423	— 35	—0,7364	— 30	—0,1746	488
—3,30	0,0389	—269	—1,3458	— 16	—0,7394	— 21	—0,1258	487
—3,28	0,0120	—269	—1,3474	+ 1	—0,7415	— 10	—0,0771	486
—3,26	—0,0149	—270	—1,3473	18	—0,7425	— 1	—0,0285	483
—3,24	—0,0419	—269	—1,3455	36	—0,7426	+ 9	0,0198	479
—3,22	—0,0688	—267	—1,3419	52	—0,7417	18	0,0677	476
—3,20	—0,0955	—267	—1,3367	70	—0,7399	28	0,1153	471
—3,18	—0,1222	—265	—1,8297	86	—0,7371	37	0,1624	466
—3,16	—0,1487	—263	—1,3211	102	—0,7334	46	0,2090	461
—3,14	—0,1750	—261	—1,3109	118	—0,7288	56	0,2551	454
—3,12	—0,2011	—259	—1,2991	133	—0,7232	65	0,3005	448
—3,10	—0,2270	—256	—1,2858	148	—0,7167	73	0,3453	441
—3,08	—0,2526	—252	—1,2710	163	—0,7094	82	0,3894	433
—3,06	—0,2778	—250	—1,2547	177	—0,7012	91	0,4327	425
—3,04	—0,3028	—245	—1,2370	191	—0,6921	99	0,4752	416
—3,02	—0,3273	—242	—1,2179	204	—0,6822	108	0,5168	408
—3,00	—0,3515	—237	—1,1975	217	—0,6714	115	0,5576	398

165

Pag 482

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—3,00	—0,3515	—237	—1,1975	217	—0,6714	115	0,5576	398
—2,98	—0,3752	—233	—1,1758	230	—0,6599	124	0,5974	388
—2,96	—0,3985	—228	—1,1528	242	—0,6475	131	0,6362	379
—2,94	—0,4213	—223	—1,1286	253	—0,6344	138	0,6741	367
—2,92	—0,4436	—218	—1,1033	265	—0,6206	146	0,7108	357
—2,90	—0,4654	—213	—1,0768	275	—0,6060	153	0,7465	346
—2,88	—0,4867	—207	—1,0493	286	—0,5907	159	0,7811	335
—2,86	—0,5074	—201	—1,0207	294	—0,5748	166	0,8146	323
—2,84	—0,5275	—195	—0,9913	305	—0,5582	173	0,8469	311
—2,82	—0,5470	—189	—0,9608	312	—0,5409	179	0,8780	299
—2,80	—0,5659	—183	—0,9296	321	—0,5230	184	0,9079	286
—2,78	—0,5842	—176	—0,8975	329	—0,5046	190	0,9365	275
—2,76	—0,6018	—170	—0,8646	336	—0,4856	195	0,9640	262
—2,74	—0,6188	—163	—0,8310	342	—0,4661	201	0,9902	249
—2,72	—0,6351	—156	—0,7968	348	—0,4460	205	1,0151	236
—2,70	—0,6507	—148	—0,7620	354	—0,4255	210	1,0387	223
—2,68	—0,6655	—142	—0,7266	360	—0,4045	215	1,0610	210
—2,66	—0,6797	—135	—0,6906	363	—0,3830	218	1,0820	198
—2,64	—0,6932	—127	—0,6543	368	—0,3612	222	1,1018	184
—2,62	—0,7059	—120	—0,6175	372	—0,3390	226	1,1202	171
—2,60	—0,7179	—112	—0,5803	375	—0,3164	229	1,1373	158
—2,58	—0,7291	—105	—0,5428	377	—0,2935	232	1,1531	145
—2,56	—0,7396	— 97	—0,5051	380	—0,2703	235	1,1676	132
—2,54	—0,7493	— 90	—0,4671	381	—0,2468	237	1,1808	118
—2,52	—0,7583	— 81	—0,4290	383	—0,2231	240	1,1926	106
—2,50	—0,7664	— 75	—0,3907	384	—0,1991	242	1,2032	94

166

Page 483

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—2,50	—0,7664	—75	—0,3907	384	—0,1991	242	1,2032	94
—2,48	—0,7739	—66	—0,3523	384	—0,1749	243	1,2126	80
—2,46	—0,7805	—59	—0,3139	384	—0,1506	245	1,2206	68
—2,44	—0,7864	—52	—0,2755	383	—0,1261	246	1,2274	55
—2,42	—0,7916	—43	—0,2372	383	—0,1015	247	1,2329	43
—2,40	—0,7959	—36	—0,1989	381	—0,0768	248	1,2372	31
—2,38	—0,7995	—29	—0,1608	380	—0,0520	248	1,2403	19
—2,36	—0,8024	—20	—0,1228	377	—0,0272	248	1,2422	6
—2,34	—0,8044	—14	—0,0851	376	—0,0024	249	1,2428	— 4
—2,32	—0,8058	— 5	—0,0475	372	0,0225	248	1,2424	— 16
—2,30	—0,8063	+ 1	—0,0103	369	0,0473	248	1,2408	— 28
—2,28	—0,8062	9	0,0266	366	0,0721	248	1,2380	— 38
—2,26	—0,8053	16	0,0632	362	0,0969	246	1,2342	— 49
—2,24	—0,8037	24	0,0994	358	0,1215	245	1,2293	— 60
—2,22	—0,8013	31	0,1352	354	0,1460	244	1,2233	— 70
—2,20	—0,7982	37	0,1706	348	0,1704	243	1,2163	— 80
—2,18	—0,7945	45	0,2054	344	0,1947	240	1,2083	— 89
—2,16	—0,7900	51	0,2398	339	0,2187	239	1,1994	— 99
—2,14	—0,7849	58	0,2737	333	0,2426	237	1,1895	—109
—2,12	—0,7791	65	0,3070	328	0,2663	235	1,1786	—117
—2,10	—0,7726	71	0,3398	321	0,2898	232	1,1669	—126
—2,08	—0,7655	78	0,3719	315	0,3130	229	1,1543	—134
—2,06	—0,7577	83	0,4034	309	0,3359	227	1,1409	—143
—2,04	—0,7494	90	0,4343	303	0,3586	224	1,1266	—150
—2,02	—0,7404	96	0,4646	296	0,3810	221	1,1116	—158
—2,00	—0,7308	102	0,4942	288	0,4031	217	1,0958	—164

167

Page 484

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—2,00	—0,7308	102	0,4942	288	0,4031	217	1,0958	—164
—1,98	—0,7206	107	0,5230	282	0,4248	214	1,0794	—172
—1,96	—0,7099	113	0,5512	275	0,4462	211	1,0622	—178
—1,94	—0,6986	119	0,5787	267	0,4673	207	1,0444	—184
—1,92	—0,6867	123	0,6054	260	0,4880	203	1,0260	—191
—1,90	—0,6744	129	0,6314	253	0,5083	200	1,0069	—196
—1,88	—0,6615	134	0,6567	245	0,5283	195	0,9873	—201
—1,86	—0,6481	139	0,6812	237	0,5478	192	0,9672	—206
—1,84	—0,6342	143	0,7049	229	0,5670	187	0,9466	—211
—1,82	—0,6199	148	0,7278	222	0,5857	183	0,9255	—216
—1,80	—0,6051	152	0,7500	214	0,6040	178	0,9039	—219
—1,78	—0,5899	156	0,7714	206	0,6218	175	0,8820	—229
—1,76	—0,5743	161	0,7920	198	0,6393	169	0,8591	—221
—1,74	—0,5582	164	0,8118	191	0,6562	165	0,8370	—230
—1,72	—0,5418	168	0,8309	182	0,6727	161	0,8140	—233
—1,70	—0,5250	172	0,8491	175	0,6888	156	0,7907	—235
—1,68	—0,5078	175	0,8666	166	0,7044	151	0,7672	—238
—1,66	—0,4903	178	0,8832	159	0,7195	146	0,7434	—240
—1,64	—0,4725	181	0,8991	151	0,7341	142	0,7194	—241
—1,62	—0,4544	184	0,9142	144	0,7483	136	0,6953	—244
—1,60	—0,4360	187	0,9286	135	0,7619	132	0,6709	—244
—1,58	—0,4173	190	0,9421	128	0,7751	127	0,6465	—245
—1,56	—0,3983	192	0,9549	121	0,7878	122	0,6220	—246
—1,54	—0,3791	195	0,9670	113	0,8000	117	0,5974	—247
—1,52	—0,3596	197	0,9783	106	0,8117	112	0,5727	—247
—1,50	—0,3399	198	0,9889	98	0,8229	107	0,5480	—247

168

Page 485

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—1,50	—0,3399	198	0,9889	98	0,8229	107	0,5480	—247
—1,48	—0,3201	201	0,9987	91	0,8336	102	0,5233	—246
—1,46	—0,3000	203	1,0078	84	0,8438	97	0,4987	—246
—1,44	—0,2797	204	1,0162	77	0,8535	93	0,4741	—246
—1,42	—0,2593	205	1,0239	71	0,8628	87	0,4495	—244
—1,40	—0,2388	207	1,0310	63	0,8715	83	0,4251	—244
—1,38	—0,2181	208	1,0373	57	0,8798	77	0,4007	—242
—1,36	—0,1973	209	1,0430	50	0,8875	73	0,3765	—241
—1,34	—0,1764	210	1,0480	45	0,8948	68	0,3524	—238
—1,32	—0,1554	211	1,0525	38	0,9016	64	0,3286	—238
—1,30	—0,1343	212	1,0563	32	0,9080	58	0,3048	—235
—1,28	—0,1131	212	1,0595	26	0,9138	54	0,2813	—232
—1,26	—0,0919	212	1,0621	20	0,9192	50	0,2581	—231
—1,24	—0,0707	213	1,0641	15	0,9242	44	0,2350	—228
—1,22	—0,0494	214	1,0656	9	0,9286	41	0,2122	—225
—1,20	—0,0280	213	1,0665	4	0,9327	35	0,1897	—222
—1,18	—0,0067	213	1,0669	— 1	0,9362	32	0,1675	—220
—1,16	0,0146	214	1,0668	— 6	0,9394	27	0,1455	—216
—1,14	0,0360	213	1,0662	—10	0,9421	22	0,1239	—213
—1,12	0,0573	213	1,0652	—15	0,9443	19	0,1026	—210
—1,10	0,0786	212	1,0637	—20	0,9462	14	0,0816	—207
—1,08	0,0998	212	1,0617	—23	0,9476	10	0,0609	—202
—1,06	0,1210	212	1,0594	—28	0,9486	6	0,0407	—200
—1,04	0,1422	211	1,0566	—31	0,9492	2	0,0207	—195
—1,02	0,1633	210	1,0535	—35	0,9494	—1	0,0012	—192
—1,00	0,1843	210	1,0500	—39	0,9493	—6	—0,0180	—188

149

Page 486

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—1,00	0,1843	210	1,0500	—39	0,9493	— 6	—0,0180	—188
—0,98	0,2053	209	1,0461	—42	0,9487	— 9	—0,0368	—184
—0,96	0,2262	208	1,0419	—45	0,9478	—13	—0,0552	—180
—0,94	0,2470	207	1,0374	—48	0,9465	—16	—0,0732	—176
—0,92	0,2677	206	1,0326	—50	0,9449	—20	—0,0908	—172
—0,90	0,2883	205	1,0276	—53	0,9429	—24	—0,1080	—167
—0,88	0,3088	204	1,0223	—56	0,9405	—26	—0,1247	—164
—0,86	0,3292	202	1,0167	—57	0,9379	—30	—0,1411	—159
—0,84	0,3494	202	1,0110	—60	0,9349	—33	—0,1570	—155
—0,82	0,3696	200	1,0050	—62	0,9316	—36	—0,1725	—150
—0,80	0,3896	199	0,9988	—63	0,9280	—39	—0,1875	—147
—0,78	0,4095	198	0,9925	—64	0,9241	—42	—0,2022	—142
—0,76	0,4293	197	0,9861	—66	0,9199	—44	—0,2164	—137
—0,74	0,4490	195	0,9795	—67	0,9155	—48	—0,2301	—134
—0,72	0,4685	194	0,9728	—68	0,9107	—50	—0,2435	—129
—0,70	0,4879	193	0,9660	—69	0,9057	—52	—0,2564	—124
—0,68	0,5072	191	0,9591	—69	0,9005	—55	—0,2688	—121
—0,66	0,5263	189	0,9522	—70	0,8950	—58	—0,2809	—116
—0,64	0,5452	189	0,9452	—70	0,8892	—59	—0,2925	—111
—0,62	0,5641	187	0,9382	—70	0,8833	—62	—0,3036	—108
—0,60	0,5828	185	0,9312	—69	0,8771	—64	—0,3144	—103
—0,58	0,6013	184	0,9243	—70	0,8707	—66	—0,3247	— 99
—0,56	0,6197	183	0,9173	—69	0,8641	—68	—0,3346	— 94
—0,54	0,6580	182	0,9104	—69	0,8573	—69	—0,3440	— 91
—0,52	0,6562	180	0,9035	—68	0,8504	—72	—0,3531	— 86
—0,50	0,6742	178	0,8967	—66	0,8432	—73	—0,3617	— 83

Pag 487

t	u	$\Delta u'$	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
—0,50	0,6742	178	0,8967	—66	0,8432	—73	—0,3617	—83
—0,48	0,6920	178	0,8901	—66	0,8359	—75	—0,3700	—78
—0,46	0,7098	176	0,8835	—65	0,8284	—76	—0,3778	—74
—0,44	0,7274	174	0,8770	—63	0,8208	—78	—0,3852	—70
—0,42	0,7448	174	0,8707	—62	0,8130	—79	—0,3922	—67
—0,40	0,7622	172	0,8645	—60	0,8051	—80	—0,3989	—62
—0,38	0,7794	171	0,8585	—59	0,7971	—82	—0,4051	—59
—0,36	0,7965	170	0,8526	—56	0,7889	—83	—0,4110	—55
—0,34	0,8135	169	0,8470	—54	0,7806	—84	—0,4165	—51
—0,32	0,8304	168	0,8416	—52	0,7722	—84	—0,4216	—48
—0,30	0,8472	167	0,8364	—50	0,7638	—86	—0,4264	—44
—0,28	0,8639	166	0,8314	—47	0,7552	—87	—0,4308	—40
—0,26	0,8805	164	0,8267	—44	0,7465	—87	—0,4348	—37
—0,24	0,8969	164	0,8223	—42	0,7378	—88	—0,4385	—34
—0,22	0,9133	164	0,8181	—39	0,7290	—89	—0,4419	—30
—0,20	0,9297	162	0,8142	—35	0,7201	—89	—0,4449	—28
—0,18	0,9459	162	0,8107	—33	0,7112	—90	—0,4477	—24
—0,16	0,9621	161	0,8074	—29	0,7022	—90	—0,4501	—21
—0,14	0,9782	161	0,8045	—25	0,6932	—91	—0,4522	—18
—0,12	0,9943	160	0,8020	—22	0,6841	—91	—0,4540	—14
—0,10	1,0103	160	0,7998	—19	0,6750	—91	—0,4554	—13
—0,08	1,0263	159	0,7979	—14	0,6659	—91	—0,4567	— 9
—0,06	1,0422	159	0,7965	—11	0,6568	—92	—0,4576	— 6
—0,04	1,0581	159	0,7954	— 6	0,6476	—92	—0,4582	— 4
—0,02	1,0740	159	0,7948	— 2	0,6384	—91	—0,4586	— 1
0,00	1,0899	159	0,7946	+ 2	0,6293	—92	—0,4587	+ 1

Page 488

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
0,00	1,0899	159	0,7946	2	0,6293	—92	—0,4587	1
0,02	1,1058	159	0,7948	7	0,6201	—92	—0,4586	3
0,04	1,1217	159	0,7955	11	0,6109	—91	—0,4583	6
0,06	1,1376	160	0,7966	16	0,6018	—92	—0,4577	9
0,08	1,1536	160	0,7982	21	0,5926	—91	—0,4568	10
0,10	1,1696	160	0,8003	26	0,5835	—91	—0,4558	13
0,12	1,1856	161	0,8029	31	0,5744	—91	—0,4545	15
0,14	1,2017	162	0,8060	36	0,5633	—90	—0,4530	17
0,16	1,2179	162	0,8096	42	0,5563	—90	—0,4513	19
0,18	1,2341	163	0,8138	47	0,5473	—90	—0,4494	20
0,20	1,2504	164	0,8185	53	0,5383	—89	—0,4474	23
0,22	1,2668	166	0,8238	59	0,5294	—89	—0,4451	24
0,24	1,2834	166	0,8297	64	0,5205	—88	—0,4427	26
0,26	1,3000	168	0,8361	71	0,5117	—88	—0,4401	27
0,28	1,3168	169	0,8432	77	0,5029	—87	—0,4374	29
0,30	1,3337	171	0,8509	83	0,4942	—87	—0,4345	30
0,32	1,3508	173	0,8592	90	0,4855	—86	—0,4315	32
0,34	1,3681	175	0,8682	96	0,4769	—85	—0,4283	33
0,36	1,3856	176	0,8778	103	0,4684	—85	—0,4250	34
0,38	1,4032	179	0,8881	110	0,4599	—84	—0,4216	36
0,40	1,4211	181	0,8991	118	0,4515	—83	—0,4180	37
0,42	1,4392	183	0,9109	124	0,4432	—83	—0,4143	37
0,44	1,4575	186	0,9233	132	0,4349	—81	—0,4106	39
0,46	1,4761	189	0,9365	140	0,4268	—81	—0,4067	40
0,48	1,4950	192	0,9505	147	0,4187	—80	—0,4027	41
0,50	1,5142	194	0,9652	156	0,4107	—80	—0,3986	41

172

Page 489

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
0,50	1,5142	194	0,9652	156	0,4107	—80	—0,3986	41
0,52	1,5336	198	0,9808	163	0,4027	—78	—0,3945	42
0,54	1,5534	201	0,9971	172	0,3949	—78	—0,3903	43
0,56	1,5735	205	1,0143	181	0,3871	—76	—0,3860	44
0,58	1,5940	208	1,0324	189	0,3795	—76	—0,3816	44
0,60	1,6148	212	1,0513	199	0,3719	—75	—0,3772	45
0,62	1,6360	217	1,0712	207	0,3644	—74	—0,3727	46
0,64	1,6577	220	1,0919	217	0,3570	—74	—0,3681	46
0,66	1,6797	225	1,1136	227	0,3496	—72	—0,3635	46
0,68	1,7022	230	1,1363	236	0,3424	—71	—0,3589	47
0,70	1,7252	234	1,1599	247	0,3353	—71	—0,3542	47
0,72	1,7486	240	1,1846	257	0,3282	—69	—0,3495	47
0,74	1,7726	244	1,2103	268	0,3213	—68	—0,3448	48
0,76	1,7970	251	1,2371	278	0,3145	—68	—0,3400	48
0,78	1,8221	255	1,2649	290	0,3077	—67	—0,3352	48
0,80	1,8476	262	1,2939	302	0,3010	—65	—0,3304	48
0,82	1,8738	268	1,3241	313	0,2945	—65	—0,3256	48
0,84	1,9006	275	1,3554	325	0,2880	—63	—0,3208	49
0,86	1,9281	0,0280	1,3879	338	0,2817	—63	—0,3159	48
0,88	1,9561	0,0288	1,4217	351	0,2754	—62	—0,3111	49
0,90	1,9849	0,029	1,4568	364	0,2692	—61	—0,3062	48
0,92	2,014	0,031	1,4932	377	0,2631	—59	—0,3014	49
0,94	2,045	31	1,5309	392	0,2572	—59	—0,2965	48
0,96	2,076	31	1,5701	406	0,2513	—58	—0,2917	48
0,98	2,107	33	1,6107	420	0,2455	—57	—0,2869	48
1,00	2,140	34	1,6527	436	0,2398	—56	—0,2821	48

173

Pag 490

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
1,00	2,140	34	1,6527	436	0,2398	—56	—0,2821	48
1,02	2,174	34	1,6963	451	0,2342	—55	—0,2773	48
1,04	2,208	35	1,7414	467	0,2287	—54	—0,2725	47
1,06	2,243	37	1,7881	484	0,2233	—53	—0,2678	47
1,08	2,280	37	1,8365	501	0,2180	—52	—0,2631	47
1,10	2,317	38	1,8866	0,0519	0,2128	—51	—0,2584	47
1,12	2,355	39	1,9385	0,0537	0,2077	—0,0051	—0,2537	46
1,14	2,394	41	1,9922	0,056	0,2026	—0,0049	—0,2491	46
1,16	2,435	41	2,048	0,057	0,1977	—0,00484	—0,2445	46
1,18	2,476	43	2,105	60	0,19286	—0,00476	—0,2399	45
1,20	2,519	44	2,165	61	0,18810	—466	—0,2354	45
1,22	2,563	45	2,226	64	0,18344	—457	—0,2309	45
1,24	2,608	46	2,290	65	0,17887	—448	—0,2264	44
1,26	2,654	48	2,355	68	0,17439	—440	—0,2220	44
1,28	2,702	49	2,423	71	0,16999	—431	—0,2176	43
1,30	2,751	51	2,494	73	0,16568	—422	—0,2133	43
1,32	2,802	52	2,567	75	0,16146	—414	—0,2090	42
1,34	2,854	54	2,642	78	0,15732	—405	—0,2048	0,0042
1,36	2,908	55	2,720	80	0,15327	—397	—0,2006	0,0042
1,38	2,963	57	2,800	83	0,14930	—389	—0,19643	0,00410
1,40	3,020	58	2,883	86	0,14541	—381	—0,19233	0,00405
1,42	3,078	60	2,969	89	0,14160	—372	—0,18828	399
1,44	3,138	63	3,058	92	0,13788	—365	—0,18429	395
1,46	3,201	63	3,150	95	0,13423	—357	—0,18034	389
1,48	3,264	66	3,245	98	0,13066	—349	—0,17645	384
1,50	3,330	68	3,343	102	0,12717	—341	—0,17261	379

174

Page 491

u	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
1,50	3,330	68	3,343	102	0,12717	—341	—0,17261	379
1,52	3,398	70	3,445	105	0,12376	—334	—0,16882	374
1,54	3,468	72	3,550	109	0.12042	—326	—0,16508	368
1,56	3,540	75	3,659	112	0,11716	—320	—0,16140	363
1,58	3,615	76	3,771	116	0,11396	—312	—0,15777	357
1,60	3,691	79	3,887	120	0,11084	—304	—0,15420	352
1,62	3,770	81	4,007	124	0,10780	—298	—0,15068	347
1,64	3,851	84	4,131	129	0,10482	—291	—0,14721	341
1,66	3,935	87	4,260	133	0,10191	—285	—0,14380	336
1,68	4,022	89	4,393	137	0,09906	—277	—0,14044	330
1,70	4,111	92	4,530	142	0,09629	—271	—0,13714	324
1,72	4,203	95	4,672	147	0,09358	—264	—0,13390	319
1,74	4,298	98	4,819	153	0,09094	—259	—0,13071	314
1,76	4,396	101	4,972	157	0,08835	—252	—0,12757	308
1,78	4,497	104	5,129	163	0,08583	—246	—0,12449	303
1,80	4,601	108	5,292	168	0,08337	—240	—0,12146	298
1,82	4,709	111	5,460	175	0,08097	—234	—0,11848	291
1,84	4,820	114	5,635	180	0,07863	—228	—0,11557	287
1,86	4,934	118	5,815	187	0,07635	—223	—0,11270	282
1,88	5,052	122	6,002	193	0,07412	—217	—0,10988	276
1,90	5,174	126	6,195	200	0,07195	—211	—0,10712	271
1,92	5,300	130	6,395	207	0,06984	—206	—0,10441	265
1,94	5,430	134	6,602	215	0,06778	—201	—0,10176	260
1,96	5,564	139	6,817	221	0,06577	—196	—0,09916	256
1,98	5,703	143	6,038	230	0,06381	—191	—0,09660	250
2,00	5,846	147	7,268	238	0,06190	—186	—0,09410	215

175

Page 492

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
2,00	5,846	147	7,268	238	0,06190	—186	—0,09410	245
2,02	5,993	153	7,506	247	0,06004	—180	—0,09165	240
2,04	6,146	158	7,753	255	0,05824	—177	—0,08925	235
2,06	6,304	162	8,008	264	0,05647	—171	—0,08690	230
2,08	6,466	169	8,272	274	0,05476	—167	—0,08460	226
2,10	6,635	173	8,546	284	0,05309	—162	—0,08234	220
2,12	6,808	180	8,830	293	0,05147	—159	—0,08014	216
2,14	6,988	185	9,123	305	0,04988	—153	—0,07798	211
2,16	7,173	192	9,428	315	0,04835	—150	—0,07587	207
2,18	7,365	198	9,743	327	0,04685	—146	—0,07380	202
2,20	7,563	205	10,070	339	0,04539	—141	—0,07178	197
2,22	7,768	212	10,409	351	0,04398	—138	—0,06981	194
2,24	7,980	218	10,760	364	0,04260	—134	—0,06787	188
2,26	8,198	227	11,124	378	0,04126	—130	—0,06599	185
2,28	8,425	234	11,502	391	0,03996	—126	—0,06414	180
2,30	8,659	241	11,893	405	0,03870	—123	—0,06234	176
2,32	8,900	251	12,298	421	0,03747	—120	—0,06058	171
2,34	9,151	253	12,719	436	0,03627	—116	—0,05887	168
2,36	9,409	268	13,155	452	0,03511	—112	—0,05719	164
2,38	9,677	277	13,607	469	0,03399	—110	—0,05555	160
2,40	9,954	286	14,076	487	0,03289	—106	—0,05395	156
2,42	10,240	296	14,563	505	0,03183	—104	—0,05239	152
2,44	10,536	307	15,068	524	0,03079	—100	—0,05087	148
2,46	10,843	317	15,592	543	0,02979	—97	—0,04393	145
2,48	11,160	328	16,135	564	0,02882	—95	—0,04794	141
2,50	11,488	340	16,699	585	0,02787	—91	—0,04653	138

176

Page 493

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
2,50	11,488	340	16,699	585	0,02787	—91	—0,04653	138
2,52	11,828	352	17,284	608	0,02696	—89	—0,04515	134
2,54	12,180	364	17,892	630	0,02607	—87	—0,04381	131
2,56	12,544	377	18,522	0,654	0,02520	—83	—0,04250	127
2,58	12,921	390	19,176	0,680	0,02437	—82	—0,04123	124
2,60	13,311	404	19,856	0,70	0,02355	—78	—0,03999	121
2,62	13,715	419	20,56	0,73	0,02277	—77	—0,03878	118
2,64	14,134	433	21,29	76	0,02200	—74	—0,03760	114
2,66	14,567	449	22,05	79	0,02126	—0,00072	—0,03646	112
2,68	15,016	465	22,84	82	0,02054	—0,00069	—0,03534	109
2,70	15,481	482	23,66	86	0,019849	—0,000674	—0,03425	105
2,72	15,963	499	24,52	88	0,019175	—0,000654	—0,03320	103
2,74	16,462	517	25,40	92	0,018521	—633	—0,03217	100
2,76	16,979	536	26,32	0,96	0,017888	—614	—0,03117	98
2,78	17,515	556	27,28	0,99	0,017274	—594	—0,03019	94
2,80	18,071	575	28,27	1,03	0,016680	—576	—0,02925	93
2,82	18,646	0,597	29,30	1,07	0,016104	—557	—0,02832	89
2,84	19,243	0,619	30,37	1,12	0,015547	—540	—0,02743	87
2,86	19,862	0,64	31,49	1,16	0,015007	—523	—0,02656	85
2,88	20,50	0,67	32,65	1,20	0,014484	—506	—0,02571	82
2,90	21,17	69	33,85	1,25	0,013978	—489	—0,02489	80
2,92	21,86	71	35,10	1,30	0,013489	—474	—0,02409	78
2,94	22,57	74	36,40	1,36	0,013015	—459	—0,02331	75
2,96	23,31	77	37,76	1,40	0,012556	—444	—0,02256	73
2,98	24,08	80	39,16	1,47	0,012112	—429	—0,02183	71
3.00	24,88	83	40,63	1,52	0,011683	—416	—0,02112	70

177

Paog 494

t	u	Δu	u'	Δu'	v	Δv	v'	Δv'
3,00	24,88	83	40,63	1,52	0,011683	—416	—0,02112	0,00070
3,02	25,71	86	42,15	1,59	0,011267	—401	—0,02042	0,00067
3,04	26,57	89	43,74	1,64	0,010866	—389	—0,019754	0,000651
3,06	27,46	0,92	45,38	1,72	0,010477	—376	—0,019103	631
3,08	28,38	0,96	47,10	1,78	0,010101	—363	—0,018472	613
3,10	29,34	1,00	48,88	1,86	0,009738	—351	—0,017859	595
3,12	30,34	1,03	50,74	1,93	0,009387	—340	—0,017264	577
3,14	31,37	1,08	52,67	2,01	0,009047	—328	—0,016687	559
3,16	32,45	1,11	54,68	2,09	0,008719	—317	—0,016128	543
3,18	33,56	1,16	56,77	2,18	0,008402	—306	—0,015585	526
3,20	34,72	1,20	58,95	2,26	0,008096	—296	—0,015059	511
3,22	35,92	1,25	61,21	2,37	0,007800	—286	—0,014548	494
3,24	37,17	1,29	63,58	2,45	0,007514	—277	—0,014054	479
3,26	38,46	1,35	66,03	2,56	0,007237	—266	—0,013575	465
3,28	39,81	1,40	68,59	2,67	0,006971	—258	—0,013110	450
3,30	41,21	1,45	71,26	2,77	0,006713	—249	—0,012660	436
3,32	42,66	1,51	74,03	2,89	0,006464	—240	—0,012224	423
3,34	44,17	1,57	76,92	3,02	0,006224	—232	—0,011801	409
3,36	45,74	1,63	79,94	3,13	0,005992	—224	—0,011392	396
3,38	47,37	1,69	83,07	3,27	0,005768	—216	—0,010996	384
3,40	49,06	1,76	86,34	3,41	0,005552	—209	—0,010612	371
3,42	50,82	1,83	89,75	3,55	0,005343	—201	—0,010241	360
3,44	52,65	1,90	93,30	3,69	0,005142	—194	—0,009881	348
3,46	54,55	1,98	96,99	3,86	0,004948	—187	—0,009533	337
3,48	56,53	2,06	100,85	4,02	0,004761	—181	—0,009196	326
3,50	58,59	2,14	104,87	4,18	0,004580	—174	—0,008870	315

178

Pag 495

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
3,50	58,59	2,14	104,87	4,18	0,004580	—174	—0,008870	315
3,52	60,73	2,22	109,05	4,37	0,004406	—168	—0,008555	305
3,54	62,95	2,32	113,42	4,55	0,004238	—162	—0,008250	295
3,56	65,27	2,40	117,97	4,75	0,004076	—156	—0,007955	286
3,58	67,67	2,51	122,72	4,94	0,003920	—151	—0,007669	276
3,60	70,18	2,60	127,66	5,16	0,003769	—145	—0,007393	267
3,62	72,78	2,71	132,82	5,39	0,003624	—140	—0,007126	258
3,64	75,49	2,82	138,21	5,61	0,003484	—135	—0,006868	249
3,66	78,31	2,94	143,82	5,85	0,003349	—130	—0,006619	241
3,68	81,25	3,05	149,67	6,11	0,003219	—125	—0,006378	233
3,70	84,30	3,18	155,78	6,38	0,003094	—121	—0,006145	225
3,72	87,48	3,31	162,16	6,64	0,002973	—116	—0,005920	217
3,74	90,79	3,44	168,80	6,94	0,002857	—112	—0,005703	211
3,76	94,23	3,59	175,74	7,22	0,002745	—108	—0,005492	202
3,78	97,82	3,73	182,96	7,58	0,002637	—103	—0,005290	196
3,80	101,55	3,89	190,54	7,88	0,002534	—100	—0,005094	190
3,82	105,44	4,05	198,42	8,28	0,002434	—97	—0,004904	182
3,84	109,49	4,22	206,7	8,5	0,002337	—92	—0,004722	177
3,86	113,71	4,39	215,2	9,0	0,002245	—90	—0,004545	170
3,88	118,10	4,58	224,2	9,4	0,002155	—0,000085	—0,004375	164
3,90	122,68	4,77	233,6	9,8	0,002070	—0,000083	—0,004211	159
3,92	127,45	4,97	243,4	10,2	0,001987	—0,0000795	—0,004052	153
3,94	132,42	5,18	253,6	10,6	0,0019075	—0,0000765	—0,003899	148
3,96	137,60	5,39	264,2	11,2	0,0018310	—736	—0,003751	142
3,98	142,99	5,63	275,4	11,6	0,0017574	—708	—0,003609	137
4,00	148,62	5,86	287,0	12,2	0,0016866	—681	—0,003472	133

179

Page 496

t	u	Δu	u'	Δu'	1000v	1000Δv	1000v'	1000Δv'
4,00	148,62	5,86	287,0	12,2	1,6866	—681	—3,472	133
4,02	154,48	6,11	299,2	12,7	1,6185	—655	—3,339	128
4,04	160,59	6,36	311,9	13,2	1,5530	—630	—3,211	123
4,06	166,95	6,64	325,1	13,9	1,4900	—605	—3,088	119
4,08	173,59	6,93	339,0	14,5	1,4295	—583	—2,969	114
4,10	180,52	7,22	353,5	15,1	1,3712	—560	—2,855	111
4,12	187,74	7,53	368,6	15,8	1,3152	—538	—2,744	106
4,14	195,27	7,8	384,4	16,5	1,2614	—517	—2,638	103
4,16	203,1	8,2	400,9	17,3	1,2097	—497	—2,535	98
4,18	211,3	8,6	418,2	18,1	1,1600	—478	—2,437	96
4,20	219,9	8,9	436,3	18,9	1,1122	—459	—2,341	91
4,22	228,8	9,3	455,2	19,7	1,0663	—441	—2,250	89
4,24	238,1	9,7	474,9	20,7	1,0222	—424	—2,161	0,085
4,26	247,8	10,1	495,6	21,5	0,9798	—407	—2,076	0,082
4,28	257,9	10,6	517,1	22,6	0,9391	—391	—1,9944	0,0789
4,30	268,5	11,0	539,7	23,6	0,9000	—375	—1,9155	0,0759
4,32	279,5	11,5	563,3	24,7	0,8625	—361	—1,8396	732
4,34	291,0	12,0	588,0	25,9	0,8264	—346	—1,7664	703
4,36	303,0	12,6	613,9	27,0	0,7918	—332	—1,6961	678
4,38	315,6	13,1	640,9	28,3	0,7586	—319	—1,6283	652
4,40	328,7	13,6	669,2	29,6	0,7267	—307	—1,5631	627
4,42	342,3	14,3	698,8	30,9	0,6960	—294	—1,5004	604
4,44	356,6	14,9	729,7	32,4	0,6666	—282	—1,4400	580
4,46	371,5	15,6	762,1	33,9	0,6384	—271	—1,3820	559
4,48	387,1	16,3	796,0	35,5	0,6113	—259	—1,3261	537
4,50	403,4	17,0	831,5	37,2	0,5854	—250	—1,2724	517

180

Page 497

t	u	Δu	u'	$\Delta u'$	$1000v$	$1000\Delta v$	$1000v'$	$1000\Delta v'$
4,50	403,4	17,0	831,5	37,2	0,5854	—250	—1,2724	517
4,52	420,4	17,7	868,7	38,9	0,5604	—239	—1,2207	497
4,54	438,1	18,6	907,6	40,7	0,5365	—229	—1,1710	477
4,56	456,7	19,4	948,3	42,6	0,5136	—220	—1,1233	459
4,58	476,1	20,3	990,9	44,6	0,4916	—211	—1,0774	442
4,60	496,4	21,1	1035,5	46,8	0,4705	—203	—1,0332	424
4,62	517,5	22,2	1082,3	48,9	0,4502	—194	—0,9908	408
4,64	539,7	23,1	1131,2	51,3	0,4308	—186	—0,9500	392
4,66	562,8	24,2	1182,5	53,6	0,4122	—178	—0,9108	377
4,68	587,0	25,3	1236,1	56,3	0,3944	—171	—0,8731	362
4,70	612,3	26,4	1292,4	58,9	0,3773	—164	—0,8369	347
4,72	638,7	27,6	1351,3	61,7	0,3609	—157	—0,8022	344
4,74	666,3	28,9	1413,0	64,7	0,3452	—151	—0,7688	311
4,76	695,2	30,2	1477,7	67,7	0,3301	—144	—0,7367	308
4,78	725,4	31,7	1545,4	71,0	0,3157	—138	—0,7059	296
4,80	757,1	33,0	1616,4	74,4	0,3019	—132	—0,6763	284
4,82	790,1	34,6	1690,8	78,0	0,2887	—127	—0,6479	272
4,84	824,7	36,2	1768,8	81,8	0,2760	—122	—0,6207	262
4,86	860,9	37,9	1850,6	85,6	0,2638	—116	—0,5945	251
4,88	898,8	39,6	1936,2	90,0	0,2522	—112	—0,5694	242
4,90	938,4	41,4	2026	94	0,2410	—106	—0,5452	231
4,92	979,8	43,4	2120	99	0,2304	—102	—0,5221	222
4,94	1023,2	45,4	2219	104	0,2202	—98	—0,4999	213
4,96	1068,6	47,6	2323	108	0,2104	—94	—0,4786	205
4,98	1116,2	49,7	2431	114	0,2010	—0,0090	—0,4581	196
5,00	1165,9	52,1	2545	119	0,19204	—0,00858	—0,4385	188

181

Page 498

t	u	Δu	u'	$\Delta u'$	$1000v$	$1000\Delta v$	$1000v'$	$1000\Delta v'$
5,00	1165,9	52,1	2545	119	0,19204	—0,00858	—0,4385	188
5,02	1218,0	54,5	2664	126	0,18346	—0,00822	—0,4197	180
5,04	1272,5	57,1	2790	131	0,17524	—786	—0,4017	173
5,06	1329,6	59,8	2921	138	0,16738	—752	—0,3844	166
5,08	1389,4	62,6	3059	144	0,15986	—719	—0,3678	159
5,10	1452,0	65,6	3203	152	0,15267	—689	—0,3519	153
5,12	1517,6	68,7	3355	159	0,14578	—658	—0,3366	146
5,14	1586,3	71,9	3514	167	0,13920	—630	—0,3220	140
5,16	1658,2	75,4	3681	176	0,13290	—603	—0,3080	134
5,18	1733,6	78,9	3857	184	0,12687	—576	—0,2946	129
5,20	1812,5	82,8	4041	193	0,12111	—551	—0,2817	123
5,22	1895,3	86,7	4234	203	0,11560	—527	—0,2694	118
5,24	1982,0	91	4437	213	0,11033	—504	—0,2576	113
5,26	2073	95	4650	223	0,10529	—481	—0,2463	109
5,28	2168	100	4873	235	0,10048	—461	—0,2354	104
5,30	2268	104	5108	246	0,09587	—440	—0,2250	99
5,32	2372	110	5354	259	0,09147	—420	—0,2151	0,0095
5,34	2482	115	5613	271	0,08727	—402	—0,2056	0,0092
5,36	2597	120	5884	286	0,08325	—384	—0,19644	0,00873
5,38	2717	127	6170	299	0,07941	—367	—0,18771	0,00836
5,40	2844	132	6469	315	0,07574	—351	—0,17935	801
5,42	2976	139	6784	331	0,07223	—335	—0,17134	766
5,44	3115	146	7115	347	0,06888	—320	—0,16368	735
5,46	3261	153	7462	365	0,06568	—305	—0,15635	702
5,48	3414	160	7827	384	0,06263	—292	—0,14933	671
5,50	3574	169	8211	403	0,05971	—279	—0,14262	643

Page 499

t	u	Δu	u'	$\Delta u'$	$1000v$	$1000\Delta v$	$1000v'$	$1000\Delta v'$
5,50	3574	169	8211	403	0,05971	—279	—0,14262	643
5,52	3743	176	8614	424	0,05692	—266	—0,13619	614
5,54	3919	185	9038	445	0,05426	—255	—0,13005	588
5,56	4104	194	9483	468	0,05171	—242	—0,12417	563
5,58	4298	204	9951	492	0,04929	—232	—0,11854	538
5,60	4502	214	10443	517	0,04697	—221	—0,11316	514
5,62	4716	225	10960	543	0,04476	—211	—0,10802	492
5,64	4941	236	11503	572	0,04265	—202	—0,10310	471
5,66	5177	247	12075	601	0,04063	—192	—0,09839	450
5,68	5424	260	12676	632	0,03871	—183	—0,09389	430
5,70	5684	273	13308	664	0,03688	—175	—0,08959	411
5,72	5957	286	13972	699	0,03513	—167	—0,08548	393
5,74	6243	301	14671	736	0,03346	—160	—0,08155	375
5,76	6544	316	15407	773	0,03186	—152	—0,07780	359
5,78	6860	331	16180	813	0,03034	—145	—0,07421	343
5,80	7191	349	16993	856	0,02889	—138	—0,07078	327
5,82	7540	365	17849	900	0,02751	—132	—0,06751	314
5,84	7905	385	18749	947	0,02619	—125	—0,06437	299
5,86	8290	404	19696	997	0,02494	—120	—0,06138	285
5,88	8694	424	20693	1049	0,02374	—115	—0,05853	273
5,90	9118	446	21742	1104	0,02259	—109	—0,05580	261
5,92	9564	468	22846	1162	0,02150	—103	—0,05319	218
5,94	10032	492	24008	1223	0,02047	—0,00099	—0,05071	238
5,96	10524	518	25231	1287	0,019475	—0,000944	—0,04833	227
5,98	11042	544	26518	1355	0,018531	—0,000899	—0,04606	216
6,00	11586	571	27873	1427	0,017632	—0,000857	—0,04390	207

183

Page 500

t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{6}v'$	$10^{6}\Delta v'$
6,00	11,586	0,571	27,87	1,43	17,632	—0,857	—43,90	2,07
6,02	12,157	0,601	29,30	1,50	16,775	—0,817	—41,83	1,98
6,04	12,758	632	30,80	1,58	15,958	—778	—39,85	1,88
6,06	13,390	664	32,38	1,67	15,180	—742	—37,97	1,80
6,08	14,054	698	34,05	1,75	14,438	—706	—36,17	1,71
6,10	14,752	735	35,80	1,85	13,732	—672	—34,46	1,64
6,12	15,487	772	37,65	1,95	13,060	—641	—32,82	1,56
6,14	16,259	812	39,60	2,04	12,419	—610	—31,26	1,49
6,16	17,071	855	41,64	2,16	11,809	—581	—29,77	1,42
6,18	17,926	0,898	43,80	2,28	11,228	—553	—28,35	1,36
6,20	18,824	0,946	46,08	2,39	10,675	—527	—26,99	1,29
6,22	19,770	0,99	48,47	2,53	10,148	—502	—25,70	1,23
6,24	20,76	1,05	51,00	2,66	9,646	—477	—24,47	1,18
6,26	21,81	1,10	53,66	2,80	9,169	—455	—23,29	1,12
6,28	22,91	1,16	56,46	2,96	8,714	—433	—22,17	1,07
6,30	24,07	1,22	59,42	3,11	8,281	—411	—21,10	1,02
6,32	25,29	1,28	62,53	3,28	7,870	—392	—20,08	0,97
6,34	26,57	1,35	65,81	3,46	7,478	—373	—19,113	0,926
6,36	27,92	1,42	69,27	3,65	7,105	—355	—18,187	883
6,38	29,34	1,50	72,92	3,85	6,750	—338	—17,304	840
6,40	30,84	1,58	76,77	4,05	6,412	—321	—16,464	802
6,42	32,42	1,66	80,82	4,27	6,091	—305	—15,662	763
6,44	34,08	1,76	85,09	4,51	5,786	—291	—14,899	728
6,46	35,82	1,84	89,60	4,76	5,495	—277	—14,171	693
6,48	37,66	1,94	94,36	5,01	5,218	—262	—13,478	660
6,50	39,60	2,04	99,37	5,29	4,956	—251	—12,818	629

184

Page 501

t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{5}v'$	$10^{6}\Delta v'$
6,50	39,60	2,04	99,37	5,29	4,956	—251	—12,818	629
6,52	41,64	2,15	104,66	5,57	4,705	—237	—12,189	598
6,54	43,79	2,26	110,23	5,89	4,468	—226	—11,591	571
6,56	46,05	2,38	116,12	6,20	4,242	—215	—11,020	543
6,58	48,43	2,51	122,32	6,55	4,027	—205	—10,477	517
6,60	50,94	2,65	128,87	6,91	3,822	—194	—9,960	492
6,62	53,59	2,79	135,78	7,29	3,628	—184	—9,468	469
6,64	56,38	2,94	143,07	7,69	3,444	—176	—8,999	446
6,66	59,32	3,09	150,76	8,12	3,268	—167	—8,553	425
6,68	62,41	3,26	148,88	8,56	3,101	—158	—8,128	404
6,70	65,67	3,44	167,44	9,04	2,943	—151	—7,724	385
6,72	69,11	3,63	176,48	9,55	2,792	—143	—7,339	366
6,74	72,74	3,82	186,03	10,07	2,649	—136	—6,973	349
6,76	76,56	4,02	196,10	10,6	2,513	—129	—6,624	331
6,78	80,58	4,25	206,7	11,3	2,384	—123	—6,293	315
6,80	84,83	4,48	218,0	11,8	2,261	—116	—5,978	300
6,82	89,31	4,72	229,8	12,5	2,145	— 0,111	—5,678	286
6,84	94,03	4,98	242,3	13,3	2,034	— 0,105	—5,392	271
6,86	99,01	5,24	255,6	13,9	1,9291	—0,0998	—5,121	258
6,88	104,25	5,54	269,5	14,8	1,8293	—0,0948	—4,863	246
6,90	109,79	5,84	284,3	15,5	1,7345	—899	—4,617	233
6,92	115,63	6,16	299,8	16,5	1,6446	—855	—4,384	222
6,94	121,79	6,50	316,3	17,4	1,5591	—811	—4,162	211
6,96	128,29	6,85	333,7	18,3	1,4780	—770	—3,951	201
6,98	135,14	7,24	352,0	19,4	1,4010	—731	—3,750	191
7,00	142,38	7,63	371,4	20,5	1,3279	—693	—3,559	181

185

Page 502

t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{6}v'$	$10^{6}\Delta v'$
7,00	142,38	7,63	371,4	20,5	1,3279	—693	—3,559	181
7,02	150,01	8,05	391,9	21,7	1,2586	—658	—3,378	172
7,04	158,06	8,50	413,6	22,9	1,1928	—625	—3,206	164
7,06	166,56	8,97	436,5	24,1	1,1303	—593	—3,042	156
7,08	175,53	9,46	460,6	25,6	1,0710	—562	—2,886	147
7,10	184,99	10,00	486,2	27,0	1,0148	—534	—2,739	141
7,12	194,99	10,5	513,2	28,5	0,9614	—506	—2,598	133
7,14	205,5	11,2	541,7	30,2	0,9108	—480	—2,465	127
7,16	216,7	11,7	571,9	31,9	0,8628	—455	—2,338	121
7,18	228,4	12,4	603,8	33,8	0,8173	—432	—2,217	0,114
7,20	240,8	13,1	637,6	35,6	0,7741	—410	—2,103	0,109
7,22	253,9	13,9	673,2	37,7	0,7331	—389	—1,9944	0,1031
7,24	267,8	14,6	710,9	39,9	0,6942	—368	—1,8913	0,0980
7,26	282,4	15,4	750,8	42,2	0,6574	—349	—1,7933	930
7,28	297,8	16,3	793,0	44,6	0,6225	—331	—1,7003	883
7,30	314,1	17,2	837,6	47,2	0,5894	—314	—1,6120	839
7,32	331,3	18,2	884,8	49,9	0,5580	—298	—1,5281	796
7,34	349,5	19,2	934,7	52,7	0,5282	—282	—1,4485	755
7,36	368,7	20,3	987,4	55,9	0,5000	—267	—1,3730	717
7,38	389,0	21,5	1043,3	59,1	0,4733	—254	—1,3013	681
7,40	410,5	22,7	1102,4	62,4	0,4479	—240	—1,2332	646
7,42	433,2	23,9	1164,8	66,2	0,4239	—227	—1,1686	613
7,44	457,1	25,3	1231,0	70,0	0,4012	—216	—1,1073	581
7,46	482,4	26,8	1301,0	74,0	0,3796	—204	—1,0492	552
7,48	509,2	28,3	1375,0	78,4	0,3592	—194	—0,9940	524
7,50	537,5	29,9	1453,4	82,9	0,3398	—183	—0,9416	496

Page 503

186

t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{6}v'$	$10^{6}\Delta v'$
7,50	537,5	29,9	1453,4	82,9	0,3398	—183	—0,9416	496
7,52	567,4	31,5	1536,3	87,8	0,3215	—174	—0,8920	471
7,54	598,9	33,5	1624,1	93,0	0,3041	—164	—0,8449	447
7,56	632,4	35,2	1717,1	98,3	0,2877	—156	—0,8002	423
7,58	667,6	37,4	1815,4	104,2	0,2721	—147	—0,7579	402
7,60	705,0	39,5	1919,6	110	0,2574	—140	—0,7177	381
7,62	744,5	41,7	2030	117	0,2434	—132	—0,6796	361
7,64	786,2	44,2	2147	123	0,2302	—126	—0,6435	343
7,66	830,4	46,7	2270	131	0,2176	—118	—0,6092	324
7,68	877,1	49,4	2401	139	0,2058	—0,0112	—0,5768	308
7,70	926,5	52,3	2540	147	0,19455	—0,01063	—0,5460	292
7,72	978,8	55,2	2687	155	0,18392	—1006	—0,5168	276
7,74	1034,0	58,5	2842	165	0,17386	—951	—0,4892	262
7,76	1092,5	61,9	3007	175	0,16435	—901	—0,4630	249
7,78	1154,4	65,5	3182	185	0,15534	—853	—0,4381	235
7,80	1219,9	69,2	3367	195	0,14681	—807	—0,4146	223
7,82	1289,1	73,3	3562	208	0,13874	—763	—0,3923	211
7,84	1362,4	77,6	3770	220	0,13111	—722	—0,3712	200
7,86	1440,0	82,1	3990	233	0,12389	—683	—0,3512	190
7,88	1522,1	86,9	4223	247	0,11706	—647	—0,3322	179
7,90	1609,0	92,0	4470	262	0,11059	—611	—0,3143	171
7,92	1701,0	97,4	4732	277	0,10448	—578	—0,2972	161
7,94	1798,4	103,1	5009	294	0,09870	—547	—0,2811	152
7,96	1901,5	109	5303	312	0,09323	—517	—0,2659	145
7,98	2011	115	5615	330	0,08806	—489	—0,2514	136
8,00	2126	123	5945	351	0,08317	—463	—0,2378	130

187

Page 504

t	$10^{-6}u$	$10^{-6}\Delta u$	$10^{-6}u'$	$10^{-6}\Delta u'$	$10^{9}v$	$10^{9}\Delta v$	$10^{9}v'$	$10^{9}\Delta v'$
8,00	2,126	0,123	5,945	0,351	83,17	—4,63	—237,8	13,0
8,02	2,249	0,129	6,296	0,371	78,54	—4,37	—224,8	12,2
8,04	2,378	137	6,667	394	74,17	—4,13	—212,6	11,7
8,06	2,515	146	7,061	418	70,04	—3,91	—200,9	10,9
8,08	2,661	154	7,479	443	66,13	—3,70	—189,96	10,40
8,10	2,815	163	7,922	469	62,43	—3,49	—179,56	9,83
8,12	2,978	173	8,391	498	58,94	—3,30	—169,73	9,32
8,14	3,151	183	8,889	529	55,64	—3,12	—160,41	8,81
8,16	3,334	193	9,418	560	52,52	—2,95	—151,60	8,34
8,18	3,527	206	9,978	594	49,57	—2,78	—143,26	7,89
8,20	3,733	218	10,572	631	46,79	—2,63	—135,37	7,46
8,22	3,951	230	11,203	669	44,16	—2,49	—127,91	7,07
8,24	4,181	245	11,872	710	41,67	—2,35	—120,84	6,67
8,26	4,426	259	12,582	753	39,32	—2,22	—114,17	6,33
8,28	4,685	275	13,335	800	37,10	—2,10	—107,84	5,97
8,30	4,960	291	14,135	848	35,00	—1,98	—101,87	5,65
8,32	5,251	308	14,983	0,900	33,02	[illegible],87	—96,22	5,34
8,34	5,559	327	15,883	0,956	31,15	—1,76	—90,88	5,06
8,36	5,886	347	16,839	1,014	29,39	—1,67	—85,82	4,78
8,38	6,233	368	17,853	1,076	27,72	—1,58	—81,04	4,51
8,40	6,601	390	18,929	1,14	26,14	—1,48	—76,53	4,27
8,42	6,991	413	20,07	1,22	24,66	—1,41	—72,26	4,04
8,44	7,404	439	21,29	1,28	23,25	—1,33	—68,22	3,82
8,46	7,843	465	22,57	1,37	21,92	—1,25	—64,40	3,60
8,48	8,308	493	23,94	1,45	20,67	—1,18	—60,80	3,41
8,50	8,801	523	25,39	1,54	19,492	—1,116	—57,39	3,22

188

Page 505

t	$10^{-6}u$	$10^{-6}\Delta u$	$10^{-6}u'$	$10^{-6}\Delta u'$	$10^{9}v$	$10^{9}\Delta v$	$10^{9}v'$	$10^{9}\Delta v'$
8,50	8,801	523	25,39	1,54	19,492	—1,116	—57,39	3,22
8,52	9,324	555	26,93	1,64	18,376	—1,052	—54,17	3,05
8,54	9,879	589	28,57	1,74	17,324	—0,994	—51,12	2,88
8,56	10,468	624	30,31	1,85	16,330	—0,937	—48,24	2,71
8,58	11,092	663	32,16	1,96	15,393	—0,885	—45,53	2,57
8,60	11,755	703	34,12	2,08	14,508	—0,834	—42,96	2,43
8,62	12,458	746	36,20	2,22	13,674	—0,788	—40,53	2,29
8,64	13,204	791	38,42	2,35	12,886	—0,743	—38,24	2,16
8,66	13,995	840	40,77	2,50	12,143	—0,701	—36,08	2,05
8,68	14,835	892	43,27	2,65	11,442	—0,661	—34,03	1,93
8,70	15,727	947	45,92	2,83	10,781	—0,624	—32,10	1,82
8,72	16,674	1,004	48,75	2,99	10,157	—0,588	—30,28	1,72
8,74	17,678	1,067	51,74	3,19	9,569	—0,554	—28,56	1,63
8,76	18,745	1,132	54,93	3,39	9,015	—0,524	—26,93	1,53
8,78	19,877	1,20	58,32	3,60	8,491	—0,493	—25,40	1,45
8,80	21,08	1,27	61,92	3,82	7,998	—0,465	—23,95	1,37
8,82	22,35	1,36	65,74	4,07	7,533	—0,439	—22,58	1,29
8,84	23,71	1,44	69,81	4,32	7,094	—0,413	—21,29	1,22
8,86	25,15	1,53	74,13	4,60	6,681	—0,390	—20,07	1,15
8,88	26,68	1,62	78,73	4,88	6,291	—0,367	—18,920	1,086
8,90	28,30	1,72	83,61	5,20	5,924	—0,347	—17,834	1,024
8,92	30,02	1,83	88,81	5,52	5,577	—0,326	—16,810	0,966
8,94	31,85	1,95	94,33	5,88	5,251	—0,308	—15,844	0,913
8,96	33,80	2,06	100,21	6,24	4,943	—0,290	—14,931	0,860
8,98	35,86	2,20	106,45	6,65	4,653	—0,273	—14,071	0,812
9,00	38,06		113,10		4,380		—13,259	

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PROBLEMS OF DIFFRACTION AND PROPAGATION OF ELECTROMAGNETIC WAVE--ETC(U)
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Page 506.

Addition 3.

Tables of auxiliary functions, used for calculating the current distribution.

In Chapters 2 and 4 this books are derived the formulas, which give current distribution on the surface of absolutely conducting paraboloid of rotation on which falls plane wave. According to the principle of local field established/installed in Chapter 1 in the region of penumbra, these formulas are applicable also to the currents on the surface of the convex body of arbitrary form.

The formulas indicated contain the integrals of the form

$$g(x)=\frac{1}{\sqrt{\pi}}\int_{\Gamma}\frac{e^{ixt}}{w'(t)}\,dt \tag{1.01}$$

and

$$f(x)=\frac{1}{\sqrt{\pi}}\int_{\Gamma}\frac{e^{ixt}}{w(t)}\,dt, \tag{1.02}$$

and also connected with these integrals functions

$$G(x)=e^{i\frac{x^3}{3}}g(x), \tag{1.03}$$

$$F(x)=e^{i\frac{x^3}{3}}f(x), \tag{1.04}$$

which have at the high negative values of x the simple asymptotic expressions (namely $G(x) \sim 2$; $F(x) \sim 2ix$). At the high positive values of x integrals (1.01) and (1.02) approximately are reduced to the exponential functions with the complex index, which has negative real part.

During the polarization of the incident wave, characterized by formulas (3.02) of Chapter 2, the field on the body surface is expressed only through G(x) and g(x). During another polarization [formula (4.01) Chapter 4] field expressions contain besides these functions also F(x) and f(x). It is obvious that in the general case of arbitrary polarization they are necessary and those, and other functions.

Below we give the tables of functions G(x) and F(x) for values of x from -4.5 to +1.0 through 0.1, and also the tables of functions g(x) and f(x) for values of x from -1.0 to +4.5 with the same interval. Out of these limits (when $|x| > 4,5$) are applicable with the sufficient accuracy the corresponding asymptotic expressions.

The given tables are calculated by M. G. Belkinoy and are checked L. P. Grabar', for which I express to them my deep thanks.

Page 507.

191

Table of function $G(x) = e^{i\frac{x^3}{3}} g(x)$.

x	Re G	Im G	$\lvert G\rvert$	arc G
—4,5	1,9998	—0,0055	1,9998	—9′30″
—4,4	1,9997	—0,0059	1,9997	—10′10″
—4,3	1,9997	—0,0063	1,9997	—10′50″
—4,2	1,9996	—0,0067	1,9997	—11′40″
—4,1	1,9996	—0,0073	1,9996	—12′30″
—4,0	1,9995	—0,0078	1,9995	—13′20″
—3,9	1,9994	—0,0084	1,9995	—14′30″
—3,8	1,9994	—0,0090	1,9994	—15′30″
—3,7	1,9992	—0,0098	1,9993	—16′50″
—3,6	1,9991	—0,0106	1,9991	—18′10″
—3,5	1,9990	—0,0115	1,9990	—19′40″
—3,4	1,999	—0,012	1,999	—21′
—3,3	1,999	—0,014	1,999	—23′
—3,2	1,998	—0,015	1,998	—26′
—3,1	1,998	—0,016	1,998	—28′
—3,0	1,998	—0,018	1,998	—31′
—2,9	1,997	—0,020	1,997	—34′
—2,8	1,996	—0,022	1,996	—37′
—2,7	1,996	—0,024	1,996	—41′
—2,6	1,995	—0,026	1,995	—46′
—2,5	1,993	—0,029	1,994	—51′
—2,4	1,992	—0,033	1,992	—56′
—2,3	1,990	—0,036	1,990	—1°03′
—2,2	1,988	—0,040	1,988	—1°10′
—2,1	1,985	—0,045	1,985	—1°18′
—2,0	1,981	—0,050	1,982	—1°27′
—1,9	1,977	—0,056	1,977	—1°37′

192

Page 508.

x	Re G	Im G	|G|	arc G
—1,8	1,971	—0,062	1,972	—1°47′
—1,7	1,965	—0,068	1,966	—1°58′
—1,6	1,956	—0,075	1,958	—2°11′
—1,5	1,946	—0,082	1,948	—2°25′
—1,4	1,933	—0,090	1,936	—2°40′
—1,3	1,919	—0,098	1,921	—2°55′
—1,2	1,901	—0,105	1,904	—3°10′
—1,1	1,880	—0,113	1,884	—3°27′
—1,0	1,857	—0,119	1,861	—3°40′
—0,9	1,829	—0,123	1,833	—3°51′
—0,8	1,798	—0,126	1,802	—4°00′
—0,7	1,762	—0,126	1,766	—4°05′
—0,6	1,722	—0,122	1,726	—4°03′
—0,5	1,678	—0,115	1,682	—3°54′
—0,4	1,630	—0,103	1,633	—3°36′
—0,3	1,578	—0,086	1,580	—3°06′
—0,2	1,522	—0,063	1,523	—2°22′
—0,1	1,462	—0,034	1,463	—1°21′
0	1,399	0	1,399	0°00′
0,1	1,333	0,040	1,334	1°44′
0,2	1,263	0,086	1,266	3°55′
0,3	1,189	0,137	1,197	6°35′
0,4	1,111	0,193	1,128	9°51′
0,5	1,029	0,252	1,059	13°45′
0,6	0,941	0,312	0,991	18°22′
0,7	0,846	0,373	0,924	23°47′
0,8	0,744	0,432	0,860	30°08′
0,9	0,634	0,484	0,798	37°22′
1,0	0,515	0,529	0,738	45°44′

193

Page 509.

Table of function $g(x)=e^{-i\frac{x^3}{3}}G(x)$.

x	Re g	Im g	$\|g\|$	arc g
−1,0	1,794	0,495	1,861	15°26′
−0,9	1,805	0,320	1,833	10°04′
−0,8	1,793	0,181	1,802	5°47′
−0,7	1,765	0,076	1,766	2°28′
−0,6	1,726	+0,002	1,726	+0°04′
−0,5	1,681	−0,045	1,682	−1°31′
−0,4	1,632	−0,068	1,633	−2°23′
−0,3	1,578	−0,071	1,580	−2°35′
−0,2	1,522	−0,059	1,523	−2°13′
−0,1	1,462	−0,034	1,463	−1°20′
0	1,399	0	1,399	0°00′
0,1	1,333	0,040	1,334	1°43′
0,2	1,263	0,083	1,266	3°45′
0,3	1,190	0,127	1,197	6°04′
0,4	1,115	0,169	1,128	8°37′
0,5	1,038	0,209	1,059	11°21′
0,6	0,961	0,244	0,991	14°14′
0,7	0,883	0,274	0,924	17°14′
0,8	0,806	0,299	0,860	20°19′
0,9	0,732	0,317	0,798	23°27′
1,0	0,660	0,331	0,738	26°38′
1,1	0,591	0,339	0,682	29°50′
1,2	0,527	0,343	0,628	33°02′
1,3	0,467	0,342	0,578	36°13′
1,4	0,411	0,338	0,532	39°25′
1,5	0,360	0,330	0,488	42°34′
1,6	0,313	0,320	0,448	45°42′

193

Page 509.

Table of function $g(x) = e^{-i\frac{x^3}{3}} G(x)$.

x	Re g	Im g	$\lvert g \rvert$	arc g
—1,0	1,794	0,495	1,861	15°26′
—0,9	1,805	0,320	1,833	10°04′
—0,8	1,793	0,181	1,802	5°47′
—0,7	1,765	0,076	1,766	2°28′
—0,6	1,726	+0,002	1,726	+0°04′
—0,5	1,681	—0,045	1,682	—1°31′
—0,4	1,632	—0,068	1,633	—2°23′
—0,3	1,578	—0,071	1,580	—2°35′
—0,2	1,522	—0,059	1,523	—2°13′
—0,1	1,462	—0,034	1,463	—1°20′
0	1,399	0	1,399	0°00′
0,1	1,333	0,040	1,334	1°43′
0,2	1,263	0,083	1,266	3°45′
0,3	1,190	0,127	1,197	6°04′
0,4	1,115	0,169	1,128	8°37′
0,5	1,038	0,209	1,059	11°21′
0,6	0,961	0,244	0,991	14°14′
0,7	0,883	0,274	0,924	17°14′
0,8	0,806	0,299	0,860	20°19′
0,9	0,732	0,317	0,798	23°27′
1,0	0,660	0,331	0,738	26°38′
1,1	0,591	0,339	0,682	29°5[illegible]
1,2	0,527	0,343	0,628	33°02′
1,3	0,467	0,342	0,578	36°13′
1,4	0,411	0,338	0,532	39°25′
1,5	0,360	0,330	0,488	42°34′
1,6	0,313	0,320	0,448	45°12′

194

Page 510

x	Re g	Im g	|g|	arc g
1,7	0,270	0,309	0,410	48°48′
1,8	0,232	0,296	0,376	51°53′
1,9	0,197	0,281	0,343	54°56′
2,0	0,167	0,267	0,315	57°59′
2,1	0,140	0,252	0,289	61°00′
2,2	0,116	0,237	0,264	64°00′
2,3	0,095	0,222	0,242	66°58′
2,4	0,076	0,208	0,221	69°56′
2,5	0,0596	0,1936	0,2025	72°54′
2,6	0,0453	0,1797	0,1853	75°51′
2,7	0,0330	0,1664	0,1696	78°47′
2,8	0,0224	0,1536	0,1552	81°43′
2,9	0,0133	0,1414	0,1421	84°39′
3,0	+0,0055	0,1299	0,1300	87°34′
3,1	−0,0010	0,1190	0,1190	90°30′
3,2	−0,0065	0,1088	0,1089	93°25′
3,3	−0,0110	0,0991	0,0997	96°20′
3,4	−0,0147	0,0901	0,0913	99°15′
3,5	−0,0176	0,0817	0,0836	102°10′
3,6	−0,0199	0,0739	0,0765	105°05′
3,7	−0,0216	0,0666	0,0700	108°00′
3,8	−0,0229	0,0599	0,0641	110°55′
3,9	−0,0237	0,0537	0,0587	113°50′
4,0	−0,0242	0,0480	0,0537	116°45′
4,1	−0,0244	0,0428	0,0492	119°40′
4,2	−0,0243	0,0380	0,0451	122°35′
4,3	−0,0240	0,0336	0,0413	125°30′
4,4	−0,0235	0,0296	0,0378	128°25′
4,5	−0,0228	0,0260	0,0346	131°20′

195

Page 511.

Table of function $F(x) = e^{i\frac{x^3}{3}} f(x)$.

x	Re F	Im F	$\lvert F \rvert$	arc F
—4,5	0,0261	—9,000	9,000	—89°50′
—4,4	0,0255	—8,801	8,801	—89°50′
—4,3	0,0249	—8,602	8,602	—89°50′
—4,2	0,0269	—8,401	8,401	—89°49′
—4,1	0,0287	—8,201	8,201	—89°48′
—4,0	0,0328	—8,001	8,001	—89°46′
—3,9	0,0343	—7,801	7,801	—89°45′
—3,8	0,0350	—7,601	7,601	—89°44′
—3,7	0,0363	—7,401	7,401	—89°43′
—3,6	0,0396	—7,202	7,202	—89°41′
—3,5	0,0406	—7,002	7,002	—89°40′
—3,4	0,0415	—6,802	6,802	—89°39′
—3,3	0,0442	—6,603	6,603	—89°37′
—3,2	0,0467	—6,403	6,403	—89°35′
—3,1	0,0503	—6,204	6,204	—89°32′
—3,0	0,0540	—6,004	6,004	—89°29′
—2,9	0,0575	—5,804	5,805	—89°26′
—2,8	0,0622	—5,605	5,606	—89°22′
—2,7	0,0676	—5,406	5,407	—89°17′
—2,6	0,0729	—5,207	5,208	—89°12′
—2,5	0,0787	—5,009	5,010	—89°06′
—2,4	0,0857	—4,811	4,812	—88°59′
—2,3	0,0927	—4,613	4,614	—88°51′
—2,2	0,1003	—4,416	4,417	—88°42′
—2,1	0,1093	—4,220	4,221	—88°31′
—2,0	0,1183	—4,023	4,025	—88°19′
—1,9	0,1283	—3,828	3,830	—88°05′

196

Page 512

x	Re F	Im F	$\|F\|$	arc F
—1,8	0,1386	—3,634	3,637	—87°49′
—1,7	0,1491	—3,441	3,444	—87°31′
—1,6	0,1617	—3,249	3,253	—87°09′
—1,5	0,1746	—3,059	3,064	—86°44′
—1,4	0,1890	—2,871	2,877	—86°14′
—1,3	0,204	—2,685	2,693	—85°39′
—1,2	0,221	—2,502	2,512	—84°58′
—1,1	0,238	—2,322	2,334	—84°09′
—1,0	0,256	—2,146	2,161	—83°12′
—0,9	0,274	—1,973	1,992	—82°06′
—0,8	0,292	—1,805	1,829	—80°49′
—0,7	0,310	—1,643	1,672	—79°18′
—0,6	0,327	—1,485	1,521	—77°34′
—0,5	0,344	—1,333	1,377	—75°33′
—0,4	0,358	—1,187	1,240	—73°13′
—0,3	0,371	—1,048	1,112	—70°32′
—0,2	0,380	—0,915	0,991	—67°28′
—0,1	0,386	—0,790	0,879	—63°58′
0	0,388	—0,672	0,776	—60°00′
0,1	0,386	—0,562	0,681	—55°29′
0,2	0,379	—0,458	0,595	—50°23′
0,3	0,367	—0,363	0,516	—44°39′
0,4	0,351	—0,276	0,446	—38°12′
0,5	0,328	—0,1971	0,383	—30°59′
0,6	0,302	—0,1278	0,328	—22°56′
0,7	0,271	—0,0673	0,279	—13°58′
0,8	0,235	—0,0165	0,236	—4°00′
0,9	0,1975	0,0244	0,199	7°02′
1,0	0,1577	0,0549	0,167	19°12′

197

Page 513.

Table of function $f(x) = e^{-i\frac{x^3}{3}} F(x)$.

x	Re f	Im f	$\mid f \mid$	arc f
—1,0	0,943	—1,945	2,161	—64°08′
—0,9	0,741	—1,849	1,992	—68°09′
—0,8	0,596	—1,729	1,829	—71°00′
—0,7	0,497	—1,597	1,672	—72°43′
—0,6	0,434	—1,458	1,521	—73°25′
—0,5	0,399	—1,318	1,377	—73°09′
—0,4	0,383	—1,179	1,240	—72°00′
—0,3	0,380	—1,045	1,112	—70°02′
—0,2	0,382	—0,914	0,991	—67°20′
—0,1	0,386	—0,790	0,879	—63°59′
0	0,388	—0,672	0,776	—60°00′
0,1	0,386	—0,561	0,681	—55°31′
0,2	0,378	—0,459	0,595	—50°33′
0,3	0,364	—0,366	0,516	—45°10′
0,4	0,345	—0,283	0,446	—39°25′
0,5	0,320	—0,211	0,383	—33°21′
0,6	0,291	—0,149	0,327	—27°02′
0,7	0,261	—0,0976	0,279	—20°29′
0,8	0,229	—0,0561	0,236	—13°46′
0,9	0,1977	—0,0239	0,1991	—6°53′
1,0	0,1672	0,00028	0,1672	0°06′
1,1	0,1389	0,01746	0,1400	7°10′
1,2	0,1131	0,0288	0,1167	14°18′
1,3	0,0904	0,0356	0,0971	21°29′
1,4	0,0706	0,0386	0,0805	28°10′
1,5	0,0540	0,0390	0,0666	35°53′
1,6	0,0401	0,0375	0,0549	43°04′

198

Page 514.

x	Re f	Im f	| f |	arc f
1,7	0,0289	0,0348	0,0452	50°15′
1,8	0,0200	0,0313	0,0372	57°24′
1,9	0,01312	0,0275	0,0305	64°32′
2,0	0,00788	0,0237	0,0250	71°37′
2,1	0,00403	0,0201	0,0205	78°40′
2,2	0,00126	0,01669	0,01674	85°41′
2,3	—0,00064	0,01367	0,01369	92°40′
2,4	—0,00186	0,01102	0,01118	99°36′
2,5	—0,00260	0,00875	0,00913	106°31′
2,6	—0,00296	0,00685	0,00746	113°24′
2,7	—0,00307	0,00526	0,00609	120°15′
2,8	—0,00300	0,00397	0,00497	127°05′
2,9	—0,00281	0,00292	0,00405	133°53′
3,0	—0,00256	0,00210	0,00331	140°40′
3,1	—0,00228	0,001453	0,00270	147°27′
3,2	—0,001981	0,000957	0,00220	154°12′
3,3	—0,001699	0,000587	0,001798	160°57′
3,4	—0,001433	0,000312	0,001467	167°42′
3,5	—0,001192	0,000117	0,001198	174°25′
3,6	—0,000977	—0,0000196	0,000977	181°09′
3,7	—0,000790	—0,0001092	0,000798	187°52′
3,8	—0,000630	—0,0001637	0,000651	194°34′
3,9	—0,000495	—0,0001928	0,000531	201°17′
4,0	—0,000383	—0,000204	0,000434	207°59′
4,1	—0,000291	—0,000201	0,000354	214°40′
4,2	—0,000217	—0,0001910	0,000289	221°22′
4,3	—0,0001577	—0,0001756	0,000236	228°04′
4,4	—0,0001112	—0,0001575	0,0001928	234°46′
4,5	—0,0000753	—0,0001385	0,0001576	241°28′

Pages 515-517.

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